Density Functional Theory for Fermi systems with large s-wave scattering length:

application to nuclear and atomic physics

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Seminar for postdoctoral position

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FRIB

MICHIGAN STATE UNIVERSITY

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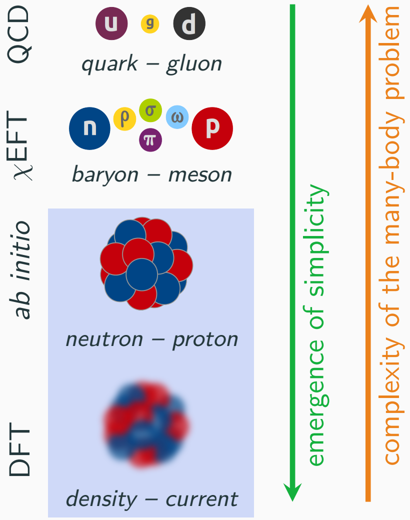
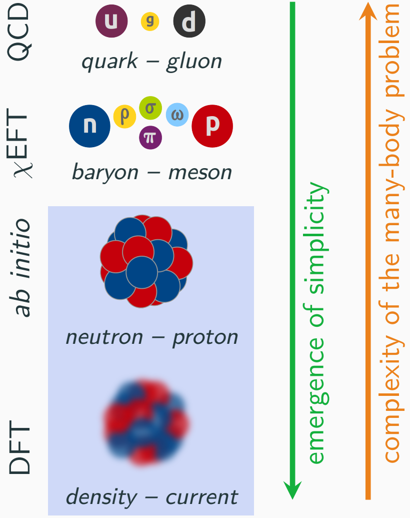
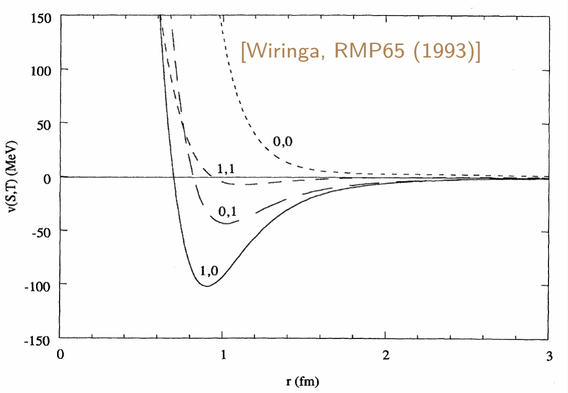

QCD
u
g
d
quark – gluon
χEFT
n
ρ
σ
ω
π
p
baryon – meson
ab initio
neutron – proton
DFT
density – current
emergence of simplicity
complexity of the many-body problem

Nuclear Physics: generalities — complexity vs simplicity





Spin $|S, S_z\rangle$

$$S = 1 \begin{cases} |1, +1\rangle & = |\uparrow\uparrow\rangle \\ |1, \ \ 0\rangle & = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |1, -1\rangle & = |\downarrow\downarrow\rangle \end{cases}$$

$$S = 0 \left\{ |0, \ \ 0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \right.$$

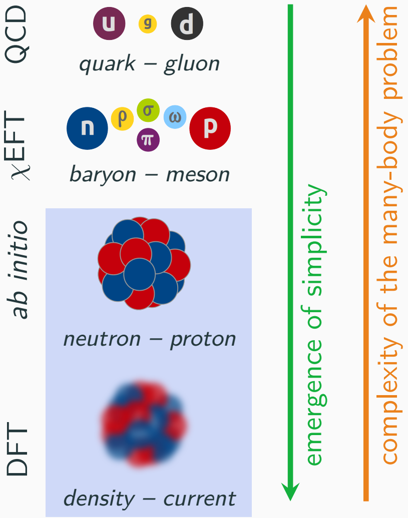
Isospin $|T, T_z\rangle$

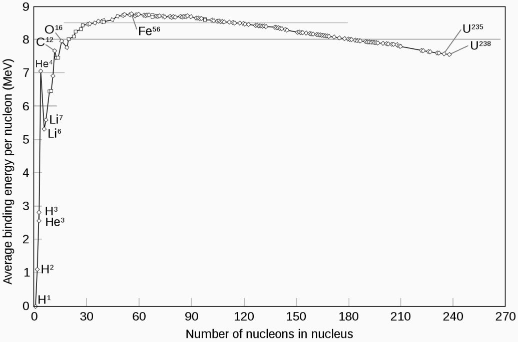
$$T = 1 \begin{cases} |1, +1\rangle & = |pp\rangle \\ |1, \ \ 0\rangle & = \frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right) \\ |1, -1\rangle & = |nn\rangle \end{cases}$$

$$T = 0 \left\{ |0, \ \ 0\rangle = \frac{1}{\sqrt{2}}\left(|pn\rangle - |np\rangle\right) \right.$$

Nuclear Physics: generalities

complexity vs simplicity





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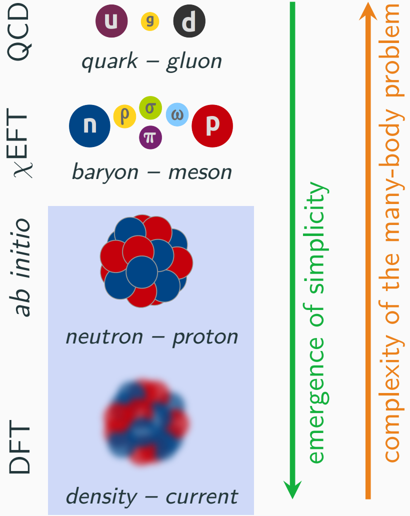

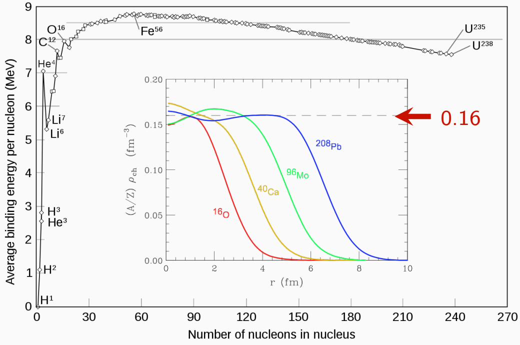

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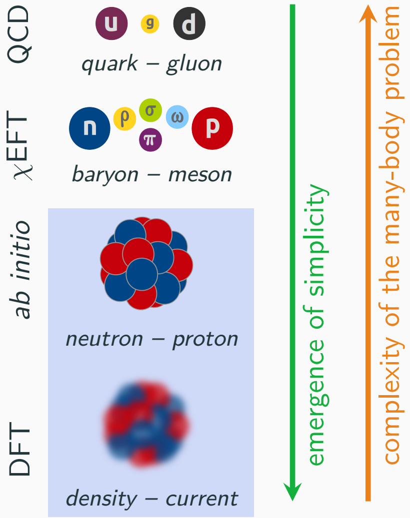
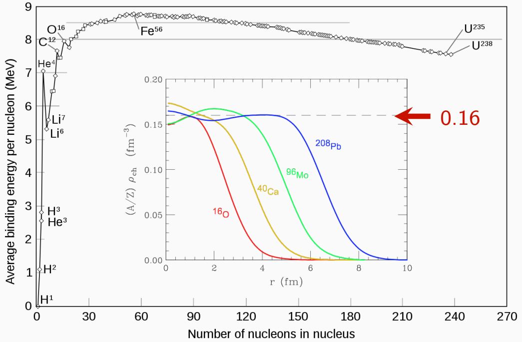
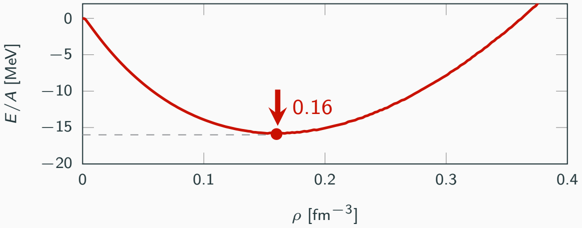
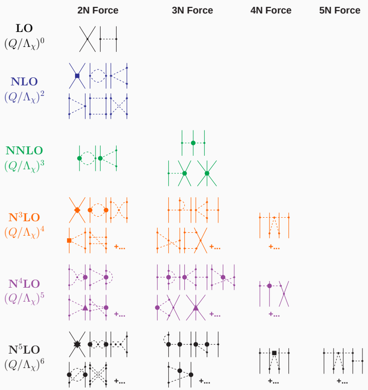






Nuclear *ab initio* methods

Starting point: χEFT
$\rightarrow$ low-energy constants



(Epelbaum, Machleidt, van Kolck, ...)

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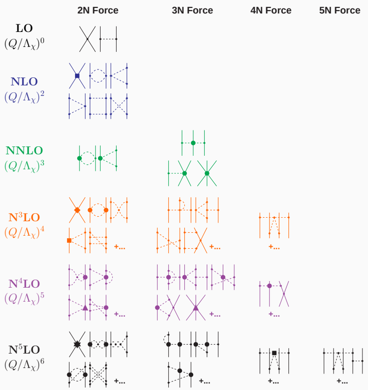

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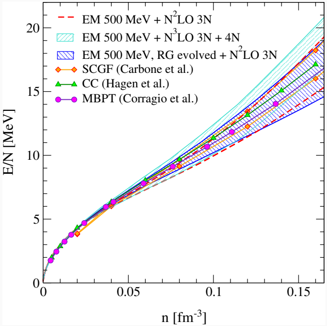
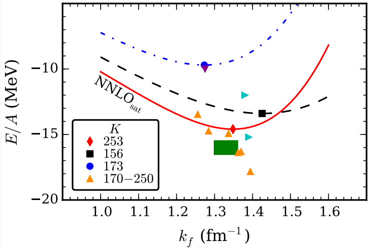


←[Hebeler et al. (2015)]
↓[Ekström et al., PRC91 (2015)]



- ✓ Systematic, Consistent, Constructive from QCD
- ∼ Exact but costly numerically (limitations)
- ✗ Errorbars for saturation points are large
- ✗ Non-explicit in terms of the LECs/density

Standard nuclear DFT

Starting point: effective interactions (Skyrme,Gogny,...)

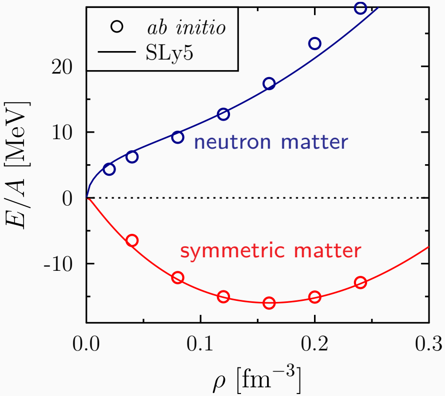
$$E = \int \mathcal{E}[\rho(\boldsymbol{r}), \nabla\rho(\boldsymbol{r}), \tau(\boldsymbol{r}), \ldots]\, d^3r$$

$\sim$ 10 parameters to be adjusted

- ✓ Correlations Beyond Mean Field
- ✓ Static, dynamic, thermo, ...
- ✓ Accurate and simple to implement

Nuclear systems $\simeq$ independent nucleons in an external one-body field

ab initio EOS:[Pandharipande et al.]



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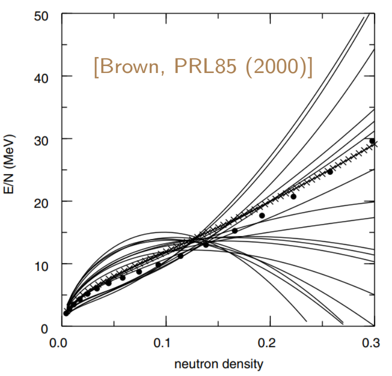

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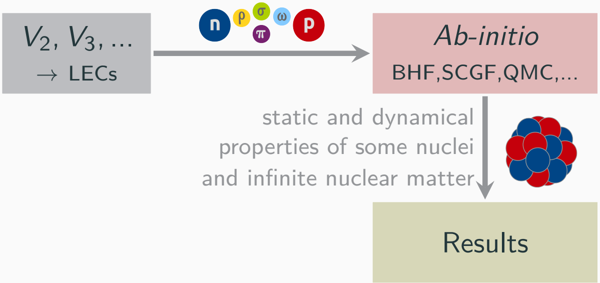
- How does the simplicity of nuclei emerge from the complexity of the nuclear interaction?
- **Can recent progress in EFT/*ab initio* help to better constrain the nuclear DFT and render it less empirical?**

Ab initio methods vs DFT picture



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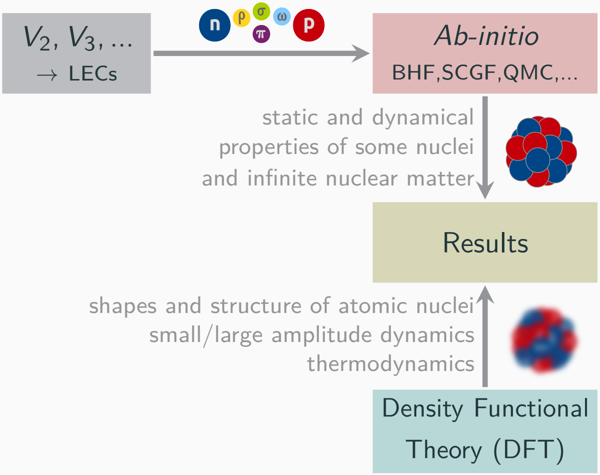

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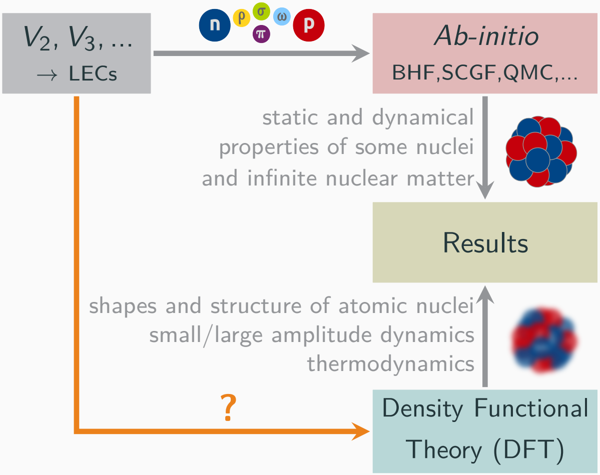

- How does the simplicity of nuclei emerge from the complexity of the nuclear interaction?
- Can recent progresses in EFT/*ab initio* help to better constraint the EDF and render it less empirical?
- **Can we directly connect the DFT parameters to the bare interaction low energy constants (LECs)?**

Dilute Fermi System: the EFT guidance

$$\langle \boldsymbol{k} | V_{\not\pi EFT} | \boldsymbol{k'} \rangle = \underbrace{C_0 + \frac{C_2}{2}\left[\boldsymbol{k}^2 + \boldsymbol{k'}^2\right]}_{s-wave} + \cdots$$

[Steele & Furnstahl, NPA762 (2000)]
[Beane et al., nucl-th/0008064 (2000)]
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$$C_0 = \frac{4\pi}{m} a_s \quad C_2 = \frac{2\pi}{m} a_s^2 r_s$$

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UV divergence properly treated
[Kaplan, Savage, Wise, NPB534 (1998)]

Lee-Yang formula $|a_s k_F| \ll 1$

$$E = E_{FG} + E^{(1)} + E^{(2)} + \cdots$$
$$= E_{FG}\left[1 + \frac{10}{9\pi}(a_s k_F) + \frac{4}{21\pi^2}(11 - 2\ln 2)(a_s k_F)^2 + \cdots\right]$$

$E_{FG} = 3k_F^2\rho/10m$ | $\rho = k_F^3/3\pi^2$

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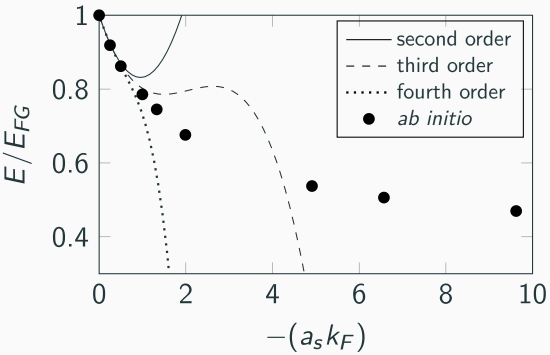

ab initio: [Carlson et al.] MBPT: [Wellenhofer et al. (2018)]

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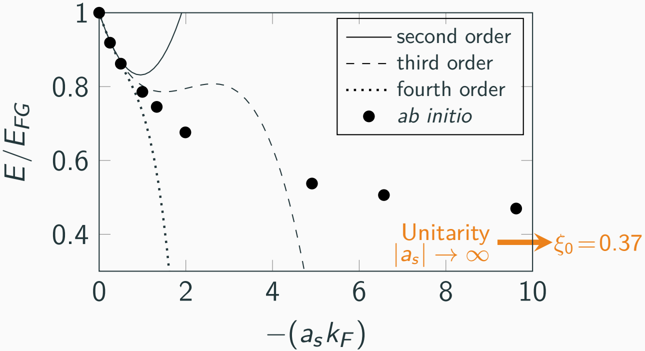

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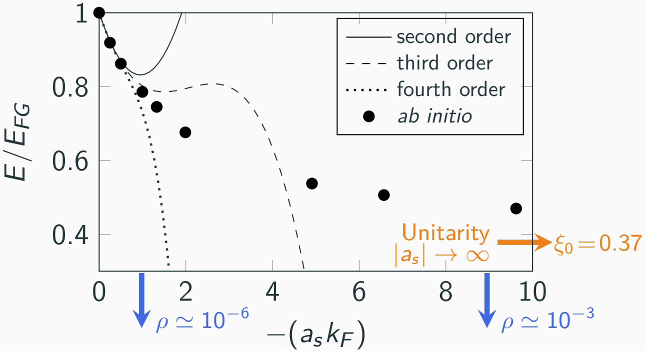

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Neutron Matter

$a_s = -18.9$ fm $\qquad r_s = 2.7$ fm

Non-empirical DFT for ultracold atom? $r_s = 0$

Low density limit as a guidance

$$\frac{E}{E_{FG}} = 1 + (a_s k_F)\gamma_1 + (a_s k_F)^2\gamma_2 + \cdots$$

Unitary limit as a guidance

$$\frac{E}{E_{FG}} \xrightarrow[|a_s|\to\infty]{} \xi_0 = 0.37 \qquad \text{(accepted value)}$$

- non-empirical $\rightarrow f(a_s)$
- DFT $\rightarrow f(\rho)$ or $f(k_F)$
- **finite limit at unitarity**

Low density limit as a guidance

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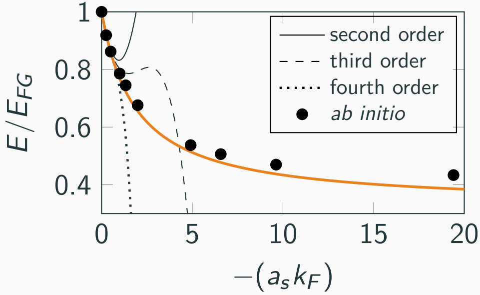

Minimal Padé approximation

$$\frac{E}{E_{FG}} = 1 + \frac{(a_s k_F)\gamma_1}{1 - (a_s k_F)\gamma_2/\gamma_1}$$

✓ valid up to second order in $(a_s k_F)$

✗ incorrect Bertsch parameter ($\simeq 0.32$)

Low density limit as a guidance

$$\frac{E}{E_{FG}} = 1 + (a_s k_F)\gamma_1 + \cancel{(a_s k_F)^2\gamma_2} + \cdots$$

Unitary limit as a guidance

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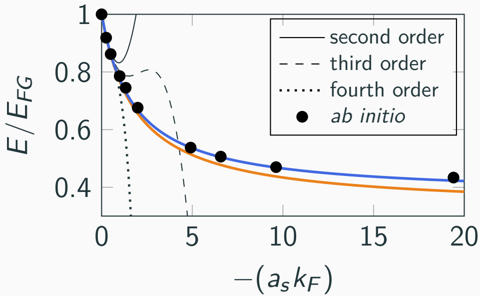

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✗ incorrect Bertsch parameter ($\simeq 0.32$)

Padé approximation + constraint

$$\frac{E}{E_{FG}} = 1 + \frac{(a_s k_F)\gamma_1}{1-(1-\xi_0)^{-1}(a_s k_F)\gamma_1}$$

✗ miss of the second order in $(a_s k_F)$

✓ exact Bertsch parameter

[Lacroix, PRA94 (2016)]

Low density limit as a guidance

$$\frac{E}{E_{FG}} = 1 + (a_s k_F)\gamma_1 + (a_s k_F)^2[\gamma_2 + (r_s k_F)\nu_1] + \cdots$$

[Fetter & Walecka book]

Non-empirical DFT including effective range $r_s \neq 0$

Low density limit as a guidance

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Unitary limit as a guidance

$$\frac{E}{E_{FG}} \xrightarrow[|a_s|\to\infty]{} \xi_0 + (r_s k_F)\eta_e + (r_s k_F)^2\delta_e + \cdots$$

[Forbes et al., PRA86 (2012)]

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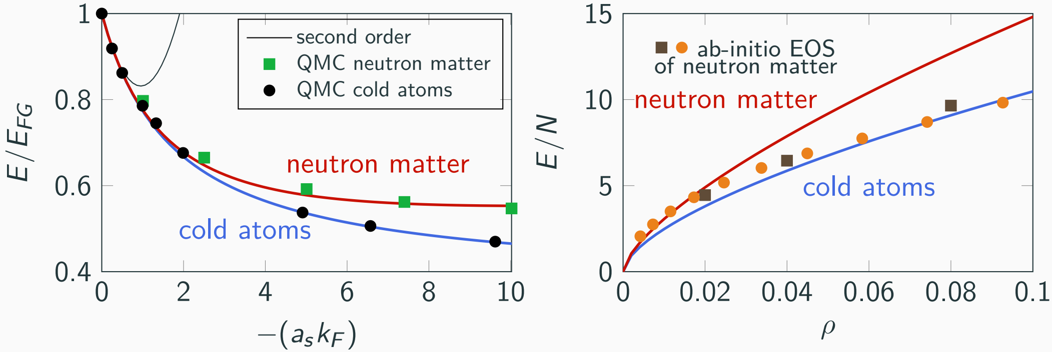
[Forbes et al., PRA86 (2012)]

Padé approximation + constraint [Lacroix, AB, et al., PRC 95 (2017)]

$$\frac{E}{E_{FG}} = 1 + \underbrace{\frac{(a_s k_F)\gamma_1}{1-(1-\xi_0)^{-1}(a_s k_F)\gamma_1}}_{\substack{\text{zero range part} \\ \to \text{ 2 parameters } \gamma_1,\xi_0}} + \underbrace{\frac{(a_s k_F)^2(r_s k_F)\nu_1 \times [1-(a_s k_F)\sqrt{\nu_1/\eta_e}]^{-1}}{1-(a_s k_F)\sqrt{\nu_1/\eta_e}+(a_s k_F)(r_s k_F)\delta_e/\eta_e}}_{\text{effective range part } \to \text{ 3 parameters } \nu_1,\eta_e,\delta_e}$$

Equation of states of dilute neutron matter



QMC: [Carlson et al.]
ab-initio EOS: [Pandharipande et al.]

Large effective range effect

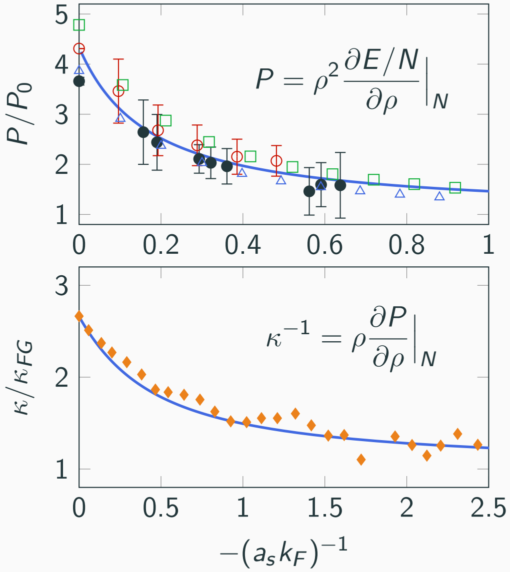
Importance of unitarity → simplicity?

Range of validity

- Lee-Yang: $\rho \lesssim 10^{-6}$ fm^{-3}
- New functional: $\rho \lesssim 10^{-2}$ fm^{-3}

[Lacroix, AB, et al., PRC 95 (2017)]

Thermodynamics of ultracold atoms



Theories

- ○ [Bulgac et al.,PRA78 (2008)]
- □ [Haussmann et al., PRA75 (2007)]
- △ [Hu et al., Europhys. Lett. 74 (2006)]

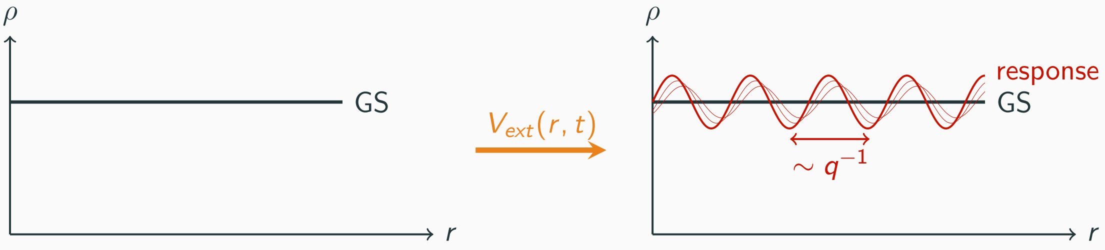
Experiments

- ● [Navon et al., Science 328 (2010)]
- ♦ [Horikoshi et al., PRX7 (2017)]

+ systematic study of effective range effect ($r_s \neq 0$)

[AB, Lacroix, PRC97 (2018)]

Linear response theory for infinite matter



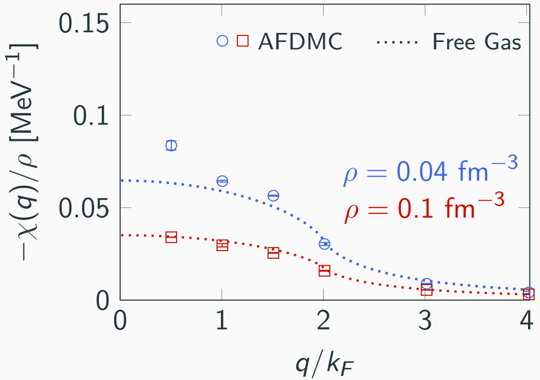
$$V_{ext} = \sum_j \phi(\boldsymbol{q},\omega) e^{i\boldsymbol{q}\cdot\boldsymbol{r}_j - i\omega t} \quad \longmapsto \quad \delta\rho = -\chi(\boldsymbol{q},\omega)\phi(\boldsymbol{q},\omega)$$

Static response function

$$\chi(q) = \lim_{\omega\to 0} \chi(q,\omega) \qquad (\text{time indep.}\, V_{ext})$$

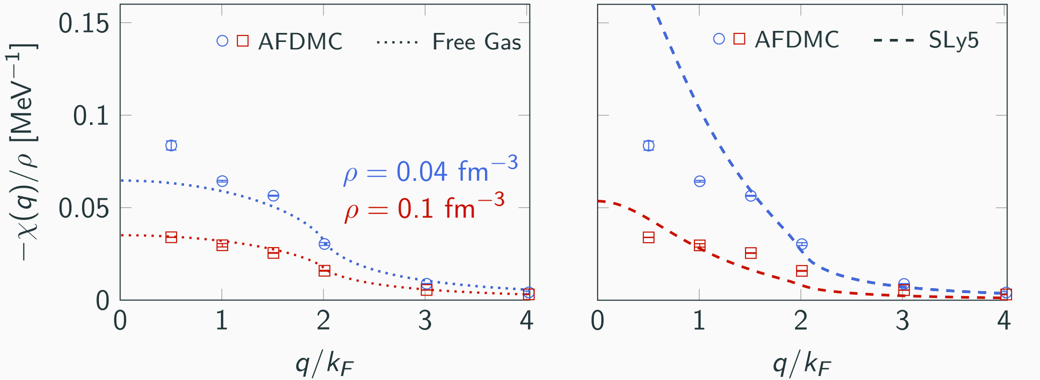
close to the response

of the Free Gas



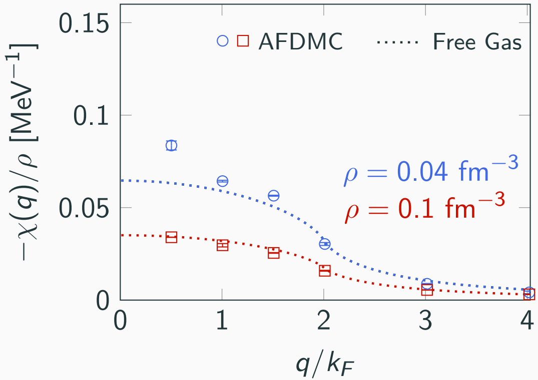
AFDMC: [Buraczynski, Gezerlis, PRC95 (2017)] close to the response of the Free Gas

standard empirical functional strongly disagree



AFDMC: [Buraczynski, Gezerlis, PRC95 (2017)]

SLy5: [Pastore et al., Phys. Rep. 563 (2015)] close to the response of the Free Gas



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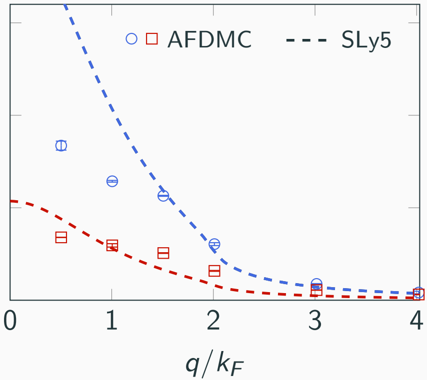

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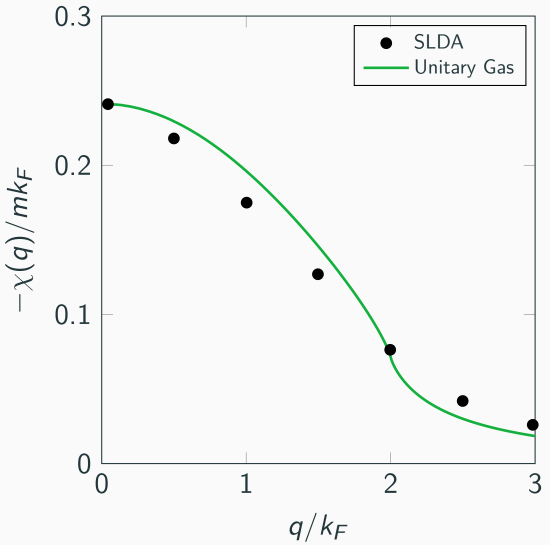
⟼ **Motivate the re-analysis using the new non-empirical functional**



Static response of Unitary Gas

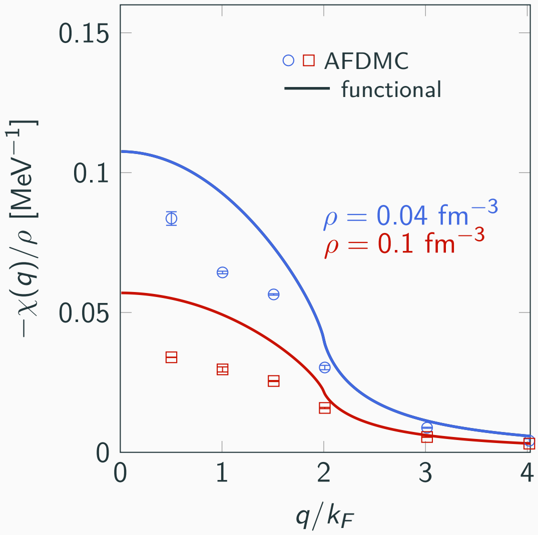
$|a_s| \to \infty$ **and** $r_s = 0$

Superfluid Local Density Approximation (SLDA) $\to$ 3 parameters

[AB, Lacroix, PRC97 (2018)]

SLDA: [Forbes & Sharma, PRA90 (2014)]

Linear response of neutron matter: effective range effect



functional: $E = E(a_s k_F, r_s k_F)$

✓ Better than standard empirical DFT

Static response of neutron matter from the non-empirical DFT

- No effective mass: $m^\star = m$

AFDMC: [Buraczynski, Gezerlis, PRC95 (2017)]
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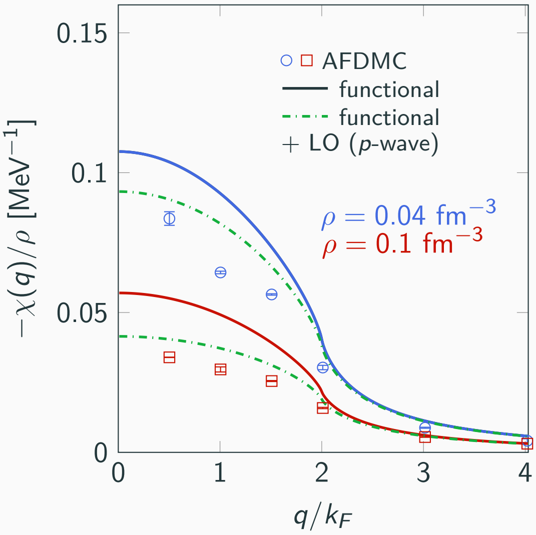

functional: $E = E(a_s k_F, r_s k_F)$

✓ Better than standard empirical DFT

Static response of neutron matter from the non-empirical DFT

- No effective mass: $m^\star = m$
- Adding the leading order of the p-wave into the functional, i.e.: $E \to E + \gamma_p (a_p k_F)^3 E_{FG}$

AFDMC match Free Gas response = compensation effect of many contributions?

AFDMC: [Buraczynski, Gezerlis, PRC95 (2017)]
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Summary

- **New non-empirical DFT linked directly to the LECs (a_s,r_s)**
  - ✓ importance of unitary limit
  - ✓ large effective range effect
  - ∼ semi-empirical approach
- **Applications: thermodynamics and linear response**
  - ✓ very promising approach
  - ✗ incomplete description $\rightarrow$ quasi-particle properties

Remains to be done in a second part

1. justify the functionals obtained *"intuitively"* starting from a more rigorous non-perturbative many-body theory
2. extend the study to the self-energy to obtain the quasi-particle properties

Basics of diagrammatic framework at zero temperature

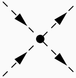
$$E = E_{FG} + E^{(1)} + E^{(2)} + \cdots$$

[Fetter & Walecka book]

$$= \frac{n_k}{\omega - e_k + i0^-} + \frac{1 - n_k}{\omega - e_k + i0^+}$$

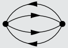
$n_k = \Theta(k_F - k)$: occupation numbers

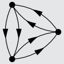
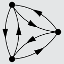
$e_k = k^2/2m$: single particle energy (FG)

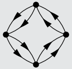
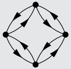
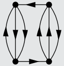
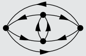
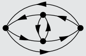
$\langle \boldsymbol{k} | V_{EFT} | \boldsymbol{k}' \rangle$  $= C_0 = 4\pi a_s/m$

Contributing energy diagrams

$E^{(1)} =$  $\rightarrow (a_s k_F) \rightarrow$ *Hartree – Fock*

$E^{(2)} =$  $\rightarrow (a_s k_F)^2 \rightarrow$ *Lee – Yang*

$E^{(3)} =$  $+$ 

$E^{(4)} =$  $+$  $+$  $+$  $+$ 

complexity

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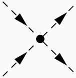
$\langle \boldsymbol{k} | V_{EFT} | \boldsymbol{k'} \rangle$  $= C_0 = 4\pi a_s / m$

Contributing energy diagrams [Ladder approximation]

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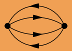
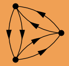
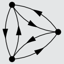
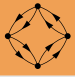
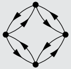
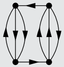
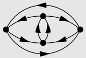
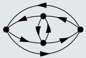
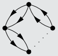
$E^{(2)} =$  $\rightarrow (a_s k_F)^2 \rightarrow$ *Lee – Yang*

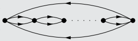
$E^{(3)} =$  $+$ 

$E^{(4)} =$  $+$  $+$  $+$  $+$ 

complexity

Ladder approximation for the energy

$$E_{int} = \sum_{n=1}^{\infty}$$  $$= \frac{80 E_{FG}}{\pi k_F^5} \int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} t dt \ \mathrm{atan} \frac{(a_s k_F)\pi I_*(s,t)}{\pi - (a_s k_F) R(s,t)}$$

$$E_{int}^{pp} = \sum_{n=1}^{\infty}$$  $$= \frac{80 E_{FG}}{\pi k_F^5} \int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} t dt \ \frac{(a_s k_F)\pi I_*(s,t)}{\pi - (a_s k_F) F(s,t)}$$

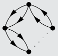
[Kaiser, NPA860 (2011)] (no pairing, no self-consistency)

$$F(s,t) = 1 + \frac{s}{k_F} - \frac{t}{k_F} \ln \left| \frac{k_F + s + t}{k_F + s - t} \right| + \frac{k_F^2 - s^2 - t^2}{2 s k_F} \ln \left| \frac{(k_F + s)^2 - t^2}{k_F - s^2 - t^2} \right|$$

$$R(s,t) = F(s,t) + F(-s,t)$$

$$I_*(s,t) = \begin{cases} t/k_F & \text{for } 0 \leq t < k_F - s \\ (k_F^2 - s^2 - t^2)/2 s k_F & \text{for } k_F - s \leq t < \sqrt{k_F^2 - s^2} \end{cases}$$

Ladder approximation for the energy I

$$E_{int} = \sum_{n=1}^{\infty}$$

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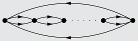
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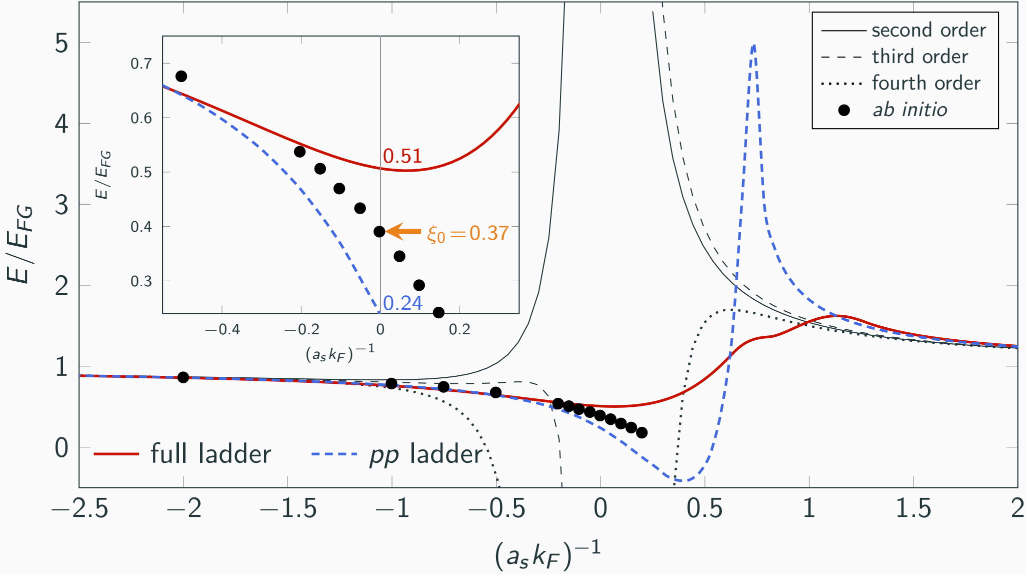
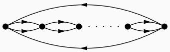
- ✓ Contain terms to all order in $(a_s k_F)$ in a compact form
- ✓ Expansion in $(a_s k_F) \rightarrow$ Lee – Yang formula
- ✓ Finite limit at unitarity ($|a_s| \rightarrow \infty$)
- ✗ Implicit function of $\rho = k_F^3 / 3\pi^2$ (goal: explicit function)

Ladder approximation for the energy



Phase-space Approximation



$$\frac{E_{pp}}{E_{FG}} = 1 + \frac{80}{\pi k_F^5} \underbrace{\int s^2 ds \int tdt}_{phase\ space} \frac{(a_s k_F)\pi I(s,t)}{1 - (a_s k_F/\pi) F(s,t)} \xrightarrow[|a_s k_F| \to \infty]{} 0.24$$

Phase-space Approximation of *pp* ladder resummation

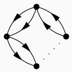
$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi} \frac{(a_s k_F)}{1 - (a_s k_F/\pi)\langle F \rangle} \xrightarrow[|a_s k_F| \to \infty]{} 0.32$$

[Heiselberg, PRA63 (2001)] [Schäfer et al., NPA762 (2005)] [Haussmann et al., PRA75 (2007)]

- ✓ Match the Lee – Yang expansion at second order
  $\langle F \rangle = \frac{6}{35}(11 - 2\ln 2)$
- ∼ More predictive near unitarity
  $\xi_0 \simeq 0.37$ (accepted value)

Adjust eventually $\langle F \rangle$ on unitary limit

- ✓ Exact at unitarity $|a_s| \to \infty$
- ✗ Lee – Yang expansion



$$\frac{E}{E_{FG}} = 1 + \frac{80}{\pi k_F^5} \underbrace{\int s^2 ds \int t dt}_{\text{phase space}} \operatorname{atan} \frac{(a_s k_F) I(s,t)}{1 - (a_s k_F/\pi) R(s,t)} \underset{|a_s k_F| \to \infty}{=} 0.51$$

Phase-space Approximation of full ladder resummation

$$\frac{E}{E_{FG}} = 1 + \frac{16}{3\pi} \operatorname{atan} \frac{5/24 (a_s k_F)}{1 - (a_s k_F/\pi) \langle R \rangle} \underset{|a_s k_F| \to \infty}{=} \mathbf{0.36}$$

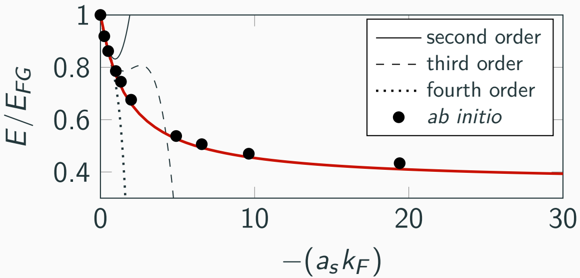
- ✓ Unitary limit well reproduced

  $\xi_0 \simeq 0.37$ (accepted value)

- ✓ Match the Lee – Yang expansion at second order

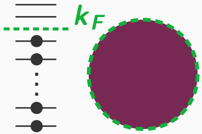
  $\langle R \rangle = \frac{6}{35}(11 - 2\ln 2)$

[AB, Lacroix, J. Phys. G **46**, (2019)]



Test particle method

$$E_{int} = \sum_{kk'} V_{eff}(k,k') n_k n_{k'}$$



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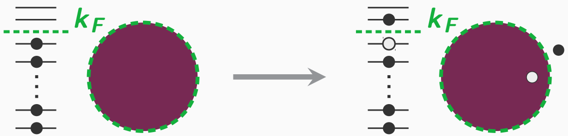
Low-lying excited states

$n_k \rightarrow n_k + \delta n_k$

$$\delta E = \sum_k \Sigma^\star(k) \delta n_k \longmapsto$$

$$\Sigma^\star(k) = U(k) + iW(k) = \frac{\delta E}{\delta n_k}$$



$\epsilon_k = \frac{k^2}{2m} + U(k)$ (single-particle energy)

$\frac{1}{2\gamma_k} = -W(k)$ (life-time)

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Close to Fermi surface $\quad v_{k_F} \equiv \partial_k \epsilon_k \big|_{k=k_F} \equiv k_F / m^\star$

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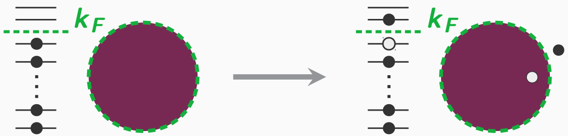

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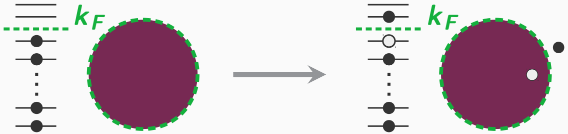

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Hugenholtz – van Hove theorem (HvH)

$$\mu = E(N+1) - E(N) = \frac{\partial E}{\partial N} = \epsilon_{k_F}$$

[Hugenholtz, Van Hove, Physica XXIV (1958)]

$$E_{int} = E_{(1)} + E_{(2)} + \cdots$$

$$E_{(1)} = \frac{10}{9\pi}(a_s k_F) E_{FG}$$

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$$\phi_2(k) \underset{k\sim k_F}{=} \frac{4}{15\pi^2}(11 - 2\ln 2) + 2\left(\frac{k}{k_F} - 1\right)\frac{8}{15\pi^2}(1 - 7\ln 2) + \cdots$$

$$\epsilon(k) = \frac{k^2}{2m} + \overbrace{Re[\Sigma^\star(k)]}^{U(k)}$$

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$$\underset{k\sim k_F}{\equiv} \boldsymbol{\mu} + (k - k_F)\frac{k_F}{\boldsymbol{m^\star}} + \cdots$$

$$\longmapsto \begin{cases} \dfrac{\mu}{\mu_{FG}} = 1 + \dfrac{4}{3\pi}(a_s k_F) + \dfrac{4}{15\pi^2}(11 - 2\ln 2)(a_s k_F)^2 + \cdots \\ \dfrac{m}{m^\star} = 1 + \dfrac{8}{15\pi^2}(1 - 7\ln 2)(a_s k_F)^2 + \cdots \end{cases}$$

Ladder approximation: analytical results

$$\Sigma^\star(k) = U(k) + iW(k) \qquad U(k<k_F) = \frac{8}{m\pi^2}\int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2-s^2}} tdt\,\mathcal{U}(s,t,k<k_F)$$

[Kaiser, EPJA49 (2013)]

- ✓ valid at low density (Galitskii formula)
- ✓ finite limit at unitarity $|a_s k_F| \to \infty$

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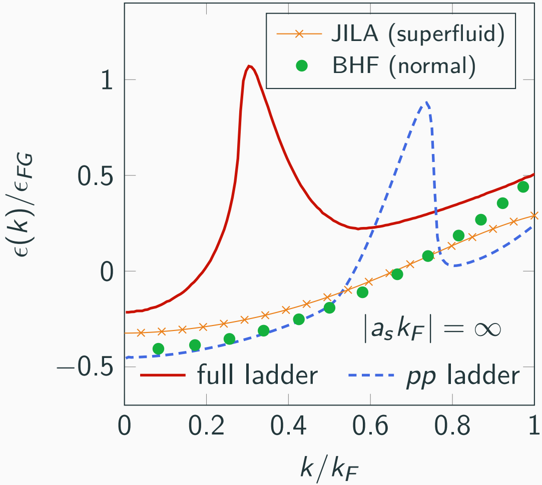

- ✓ valid at low density (Galitskii formula)
- ✓ finite limit at unitarity $|a_s k_F| \to \infty$
- ✗ bad predictivity power for $|a_s k_F| \gg 1$
- ✗ strong dependence of retained diagrams (cf. energy)

BHF: [Doggen & Kinnumen (2015)]
JILA exp.: [Stewart et al., Nature **454** (2008)]

Strategy for the self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s,t)$$

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s,t,k)$$

Strategy for the self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s,t)$$

$$\uparrow\downarrow$$

$$\mu = \left.\frac{\partial E}{\partial N}\right|_V = \epsilon(k_F)$$

$\uparrow$ $k = k_F$ HvH theorem

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s,t,k)$$

Strategy for the self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s,t)$$

PS approx. → E_{PS}

Lee − Yang

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$k = k_F$ — HvH theorem

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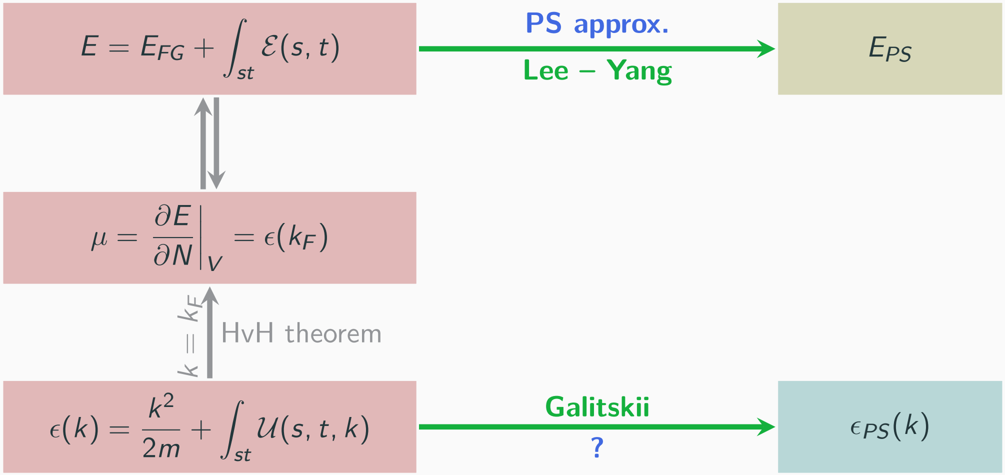
Strategy for the self-energy resummation



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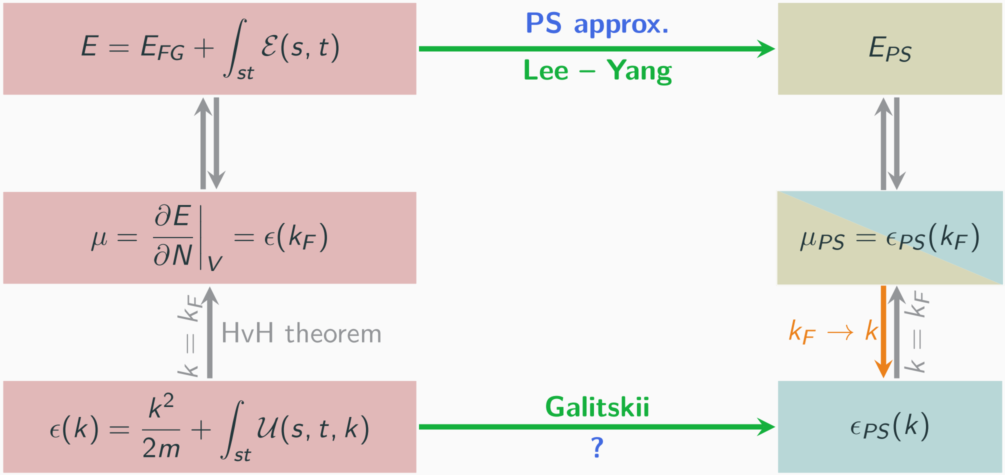

Partial phase-space average approximation

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi} \frac{(a_s k_F)}{1 - (a_s k_F/\pi)\frac{9\pi^2}{14}\phi_2(k_F)}$$

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$\mu = \left.\frac{\partial E}{\partial N}\right|_V$ ✓ Lee-Yang Formula

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$\phi_2(k_F) \to \phi_2(k)$ ✓ HvH theorem $\mu = \epsilon(k_F)$

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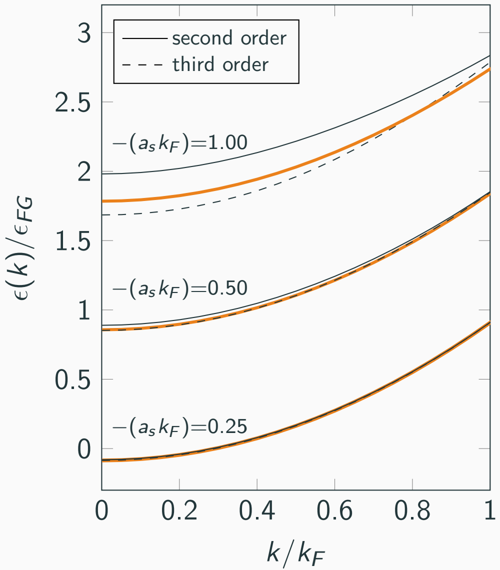
✓ Galitskii Formula

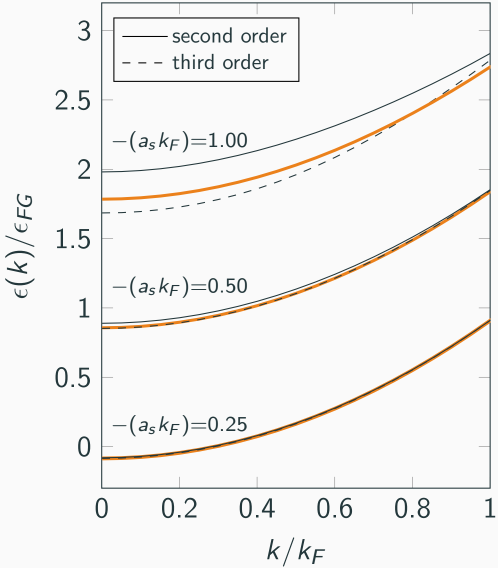
Results

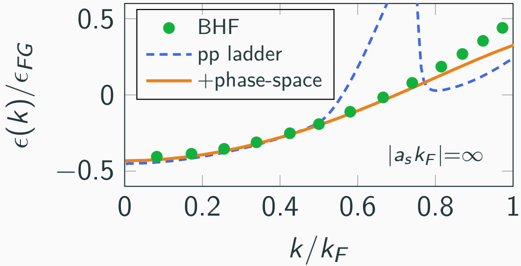


- ✓ exact expansion up to $(a_s k_F)^2$
- ✓ simpler function of the density

MBPT: [Platter et al., NPA714 (2003)]
[Doggen & Kinnumen (2015)]

Results

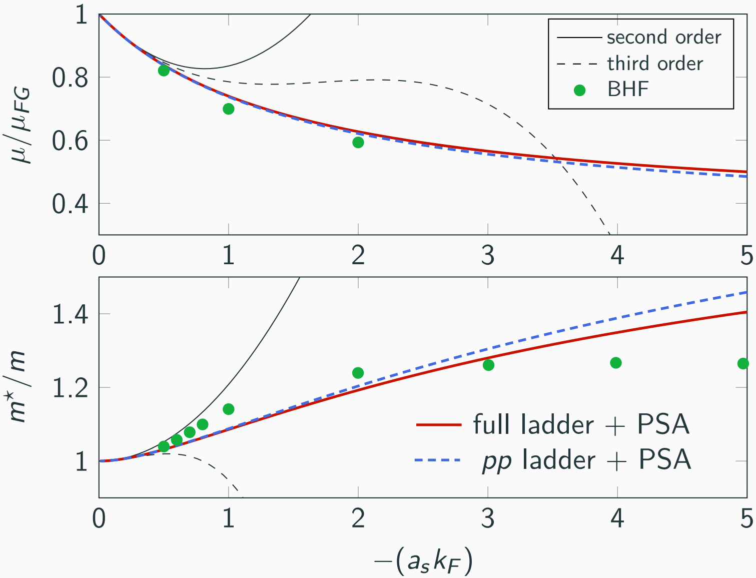




- ✓ exact expansion up to $(a_s k_F)^2$
- ✓ simpler function of the density
- ✓ pathologies removed for $|a_s k_F| \gg 1$

MBPT: [Platter et al., NPA714 (2003)]
BHF: [Doggen & Kinnumen (2015)]

Quasi-particle properties



$\mu = \epsilon(k_F)$

BHF: [Doggen & Kinnumen (2015)]
MBPT: [Platter et al., NPA714 (2003)]

$$\frac{m}{m^\star} = \frac{m}{k_F}\frac{\partial \epsilon_k}{\partial k}\bigg|_{k_F}$$

✓ Galitskii formula

✓ finite limit at unitarity

✓ beyond the perturbative regime

[AB, Lacroix, J. Phys. G **46**, (2019)]

Summary

- **non-empirical DFT**
  - ✓ **Study of the DFT as a semi-empirical function of the LECs (a_s,r_s)**
  - ✓ **Applications to cold atoms & neutron matter**
    (equation of state & thermodynamics + static response)
- **Non-perturbative resummation technique**
  - ✓ **Study at energy level → Phase-Space Approximation**
  - ✓ **link with the semi-empirical DFT = justification**
- **Study of the self-energy**
  - ✓ **generalization of the Phase-Space Approximation to the self-energy**
  - ✓ **quasi-particle properties in the non-perturbative regime**

Outlooks and perspectives

Perspectives and discussions towards non-empirical DFT

- Analytical developments with simple interactions
  - ? more realistic interaction (p-wave, ...)
  - ? superfluidity
- Cross-fertilization: DFT vs *ab initio* [Grasso, Prog. in Part. and Nucl. Phys. **106** (2019)]

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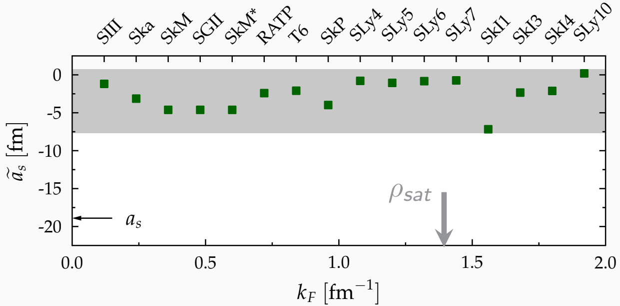
✓ **Link with the standard DFT (renormalization of the LECs)**



Skyrme functionals:
$\widetilde{a}_s(k_F) = \widetilde{a}_s = m t_0 (1 - x_0)/4\pi$

[Lacroix, AB, et al., PRC 95 (2017)]
[Furnstahl, *EFT for DFT* (2008)]

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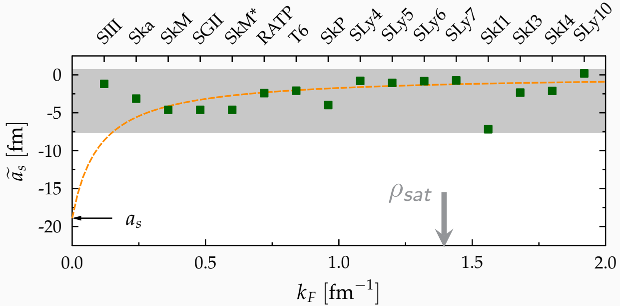

DFT re-written as:

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi}\left[\widetilde{a}_s(k_F)k_F\right]$$

Skyrme functionals:
$\widetilde{a}_s(k_F) = \widetilde{a}_s = mt_0(1 - x_0)/4\pi$

[Lacroix, AB, et al., PRC 95 (2017)]
[Furnstahl, *EFT for DFT* (2008)]

Current works and interests

- **Development towards *ab-initio* formulation of the DFT**
  - Extension of the Variational Perturbation Theory in the path integral and effective action formalisms including relevant auxiliary fields allowing for spontaneous symmetry breaking in the ground state (exponentially fast convergence with the diagrammatic truncation is expected)
  - Proof of principle (at Hartree-Fock level) with neutron drops using semi-empirical interaction $\rightarrow$ numerical implementation is ongoing [AB and S. Bogner, in prep.]
- Other interests: DFT with Legendre transforms, DME, IM-SRG, constrained MBPT, etc.