Density Functional Theory for Fermi systems with large s-wave scattering length:

application to nuclear and atomic physics

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PhD defense

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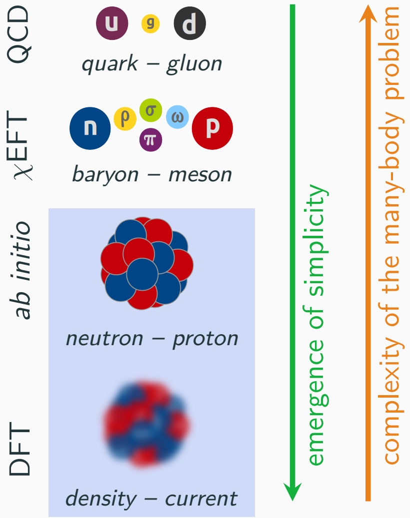
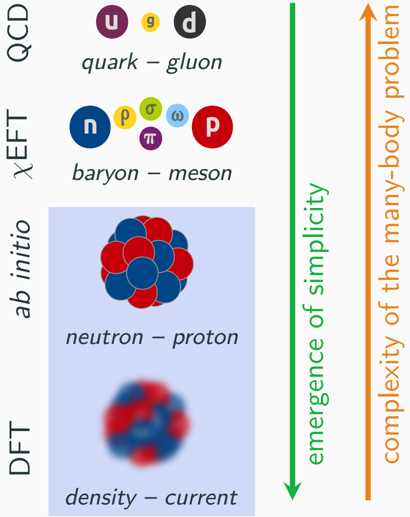
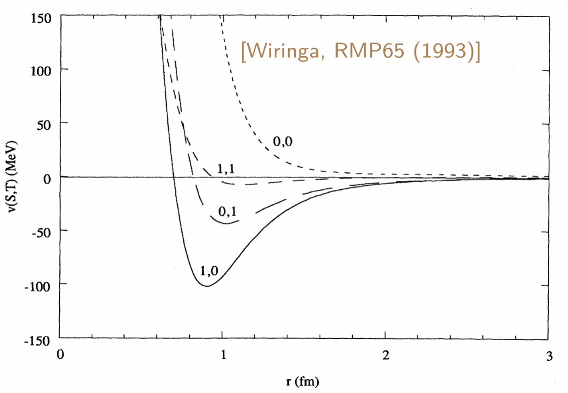
Contents


QCD
u g d
quark – gluon
χEFT
n ρ σ ω π p
baryon – meson
ab initio
neutron – proton
DFT
density – current
emergence of simplicity
complexity of the many-body problem

Nuclear Physics: generalities — complexity vs simplicity





Spin $|S, S_z\rangle$

$$S = 1 \begin{cases} |1, +1\rangle & = |\uparrow\uparrow\rangle \\ |1, \ \ 0\rangle & = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |1, -1\rangle & = |\downarrow\downarrow\rangle \end{cases}$$

$$S = 0 \left\{ |0, \ \ 0\rangle \quad = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \right.$$

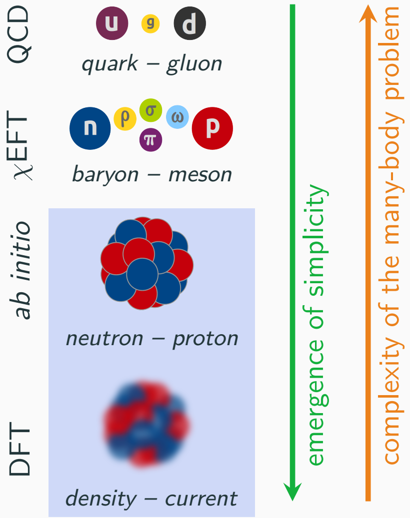
Isospin $|T, T_z\rangle$

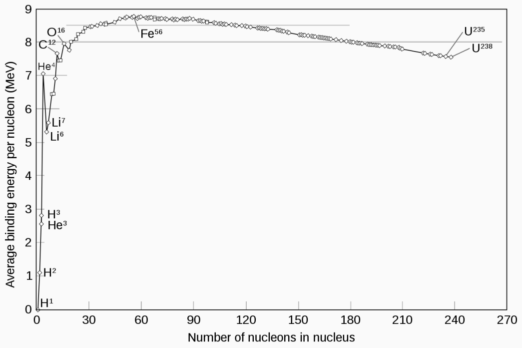
$$T = 1 \begin{cases} |1, +1\rangle & = |pp\rangle \\ |1, \ \ 0\rangle & = \frac{1}{\sqrt{2}}\left(|pn\rangle + |np\rangle\right) \\ |1, -1\rangle & = |nn\rangle \end{cases}$$

$$T = 0 \left\{ |0, \ \ 0\rangle \quad = \frac{1}{\sqrt{2}}\left(|pn\rangle - |np\rangle\right) \right.$$

Nuclear Physics: generalities

complexity vs simplicity





Nuclear Physics: generalities

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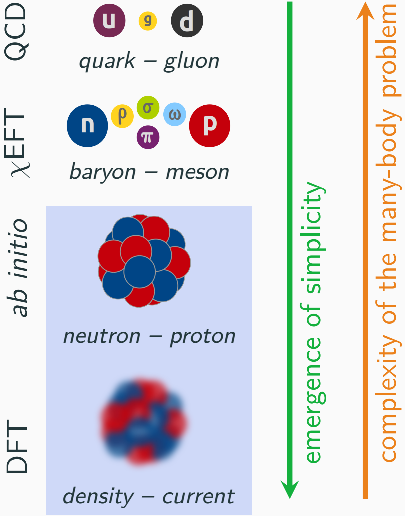

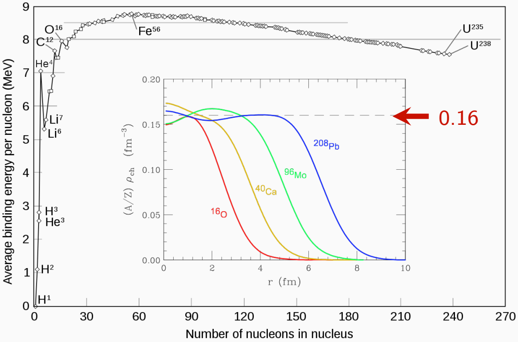

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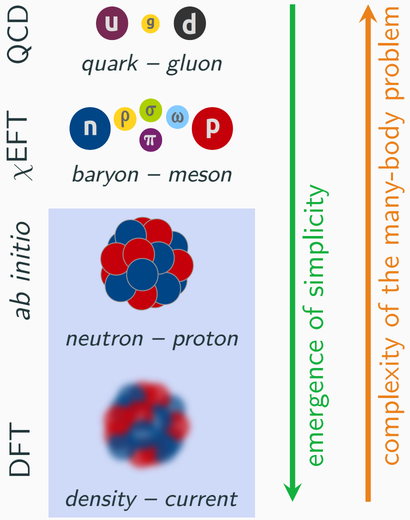
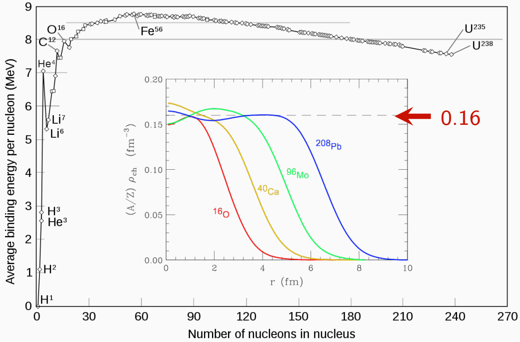
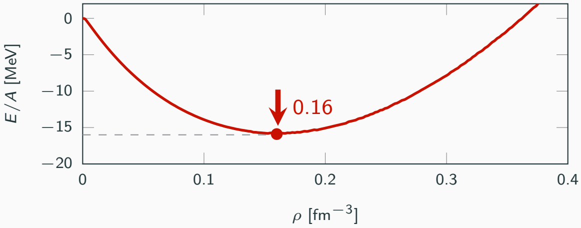
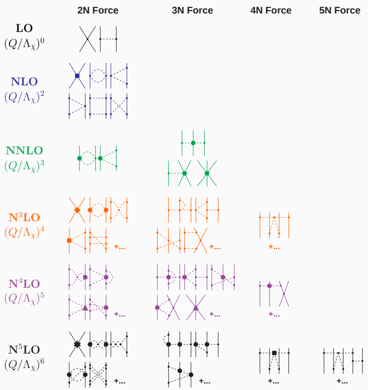






Nuclear *ab initio* methods

Starting point: χEFT
$\longrightarrow$ low-energy constants



(Epelbaum, Machleidt, van Kolck, ...)

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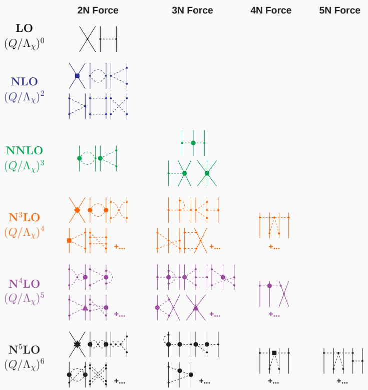

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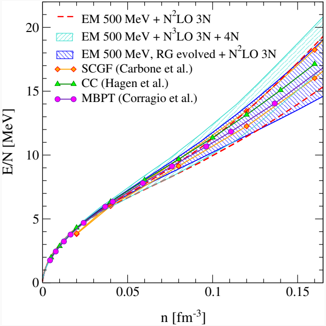
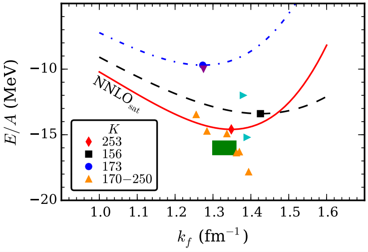


←[Hebeler et al. (2015)]

↓[Ekström et al., PRC91 (2015)]



- ✓ Systematic, Consistent, Constructive from QCD
- ∼ Exact but costly numerically (limitations)
- ✗ Errorbars for saturation points are large
- ✗ Non-explicit in terms of the LECs/density

Standard nuclear DFT

Starting point: effective interactions (Skyrme,Gogny,...)

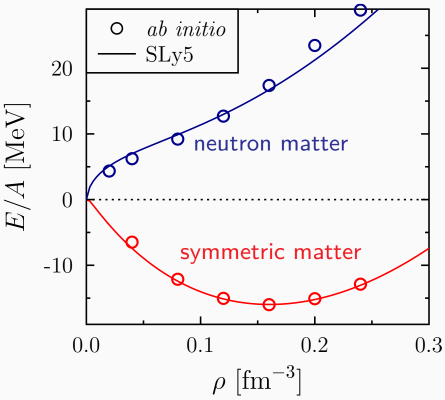
$$E = \int \mathcal{E}[\rho(\mathbf{r}), \nabla\rho(\mathbf{r}), \tau(\mathbf{r}), \dots]\, d^3r$$

$\sim$ 10 parameters to be adjusted

- ✓ Correlations Beyond Mean Field
- ✓ Static, dynamic, thermo, ...
- ✓ Accurate and simple to implement

Nuclear systems $\simeq$ independent nucleons in an external one-body field

ab initio EOS:[Pandharipande et al.]



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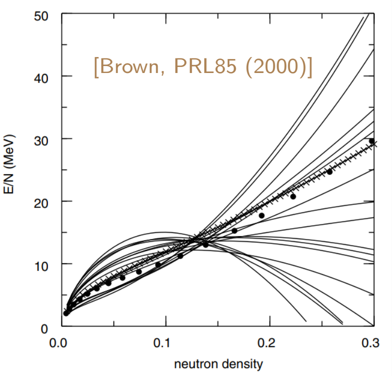

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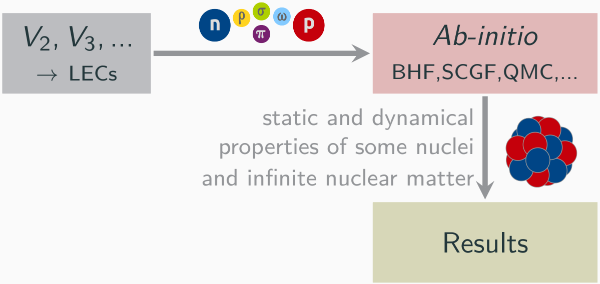
- How does the simplicity of nuclei emerge from the complexity of the nuclear interaction?
- **Can recent progress in EFT/*ab initio* help to better constrain the nuclear DFT and render it less empirical?**

Ab initio methods vs DFT picture



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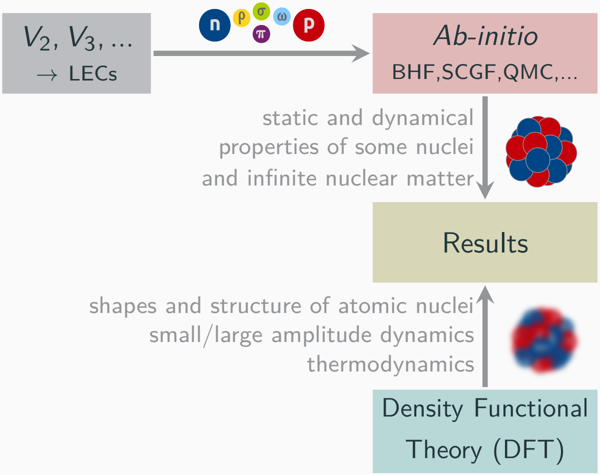

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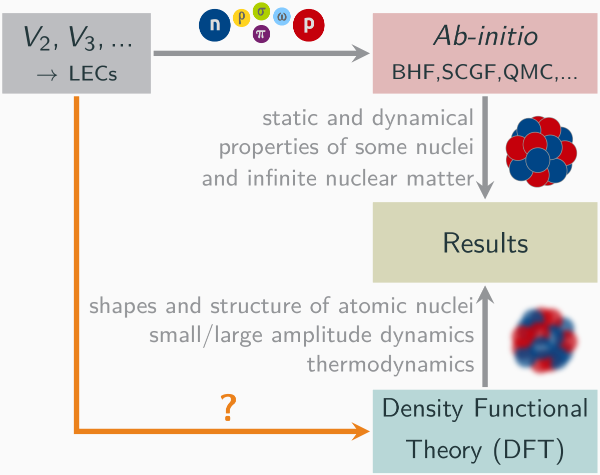

- How does the simplicity of nuclei emerge from the complexity of the nuclear interaction?
- Can recent progresses in EFT/*ab initio* help to better constraint the EDF and render it less empirical?
- **Can we directly connect the DFT parameters to the bare interaction low energy constants (LECs)?**

Dilute Fermi System: the EFT guidance

$$\langle \boldsymbol{k} | V_{\not\pi EFT} | \boldsymbol{k}' \rangle = \underbrace{C_0 + \frac{C_2}{2} \left[\boldsymbol{k}^2 + \boldsymbol{k}'^2 \right]}_{s-wave} + \cdots$$

[Steele & Furnstahl, NPA762 (2000)]
[Beane et al., nucl-th/0008064 (2000)]
[Hammer & Furnstahl, NPA678 (2000)]

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UV divergence properly treated
[Kaplan, Savage, Wise, NPB534 (1998)]

Lee-Yang formula $|a_s k_F| \ll 1$

$$E = E_{FG} + E^{(1)} + E^{(2)} + \cdots$$
$$= E_{FG}\left[1 + \frac{10}{9\pi}(a_s k_F) + \frac{4}{21\pi^2}(11 - 2\ln 2)(a_s k_F)^2 + \cdots\right]$$

$E_{FG} = 3k_F^2 \rho / 10m$ | $\rho = k_F^3 / 3\pi^2$

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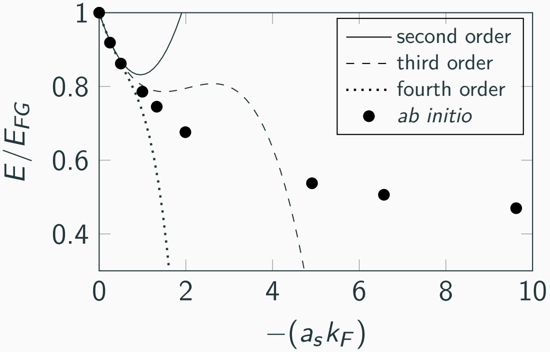

ab initio: [Carlson et al.] MBPT: [Wellenhofer et al. (2018)]

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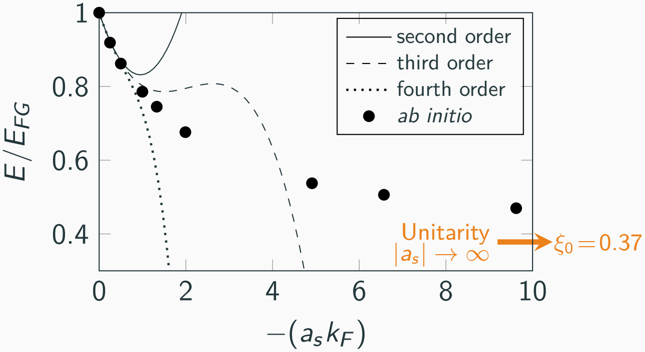

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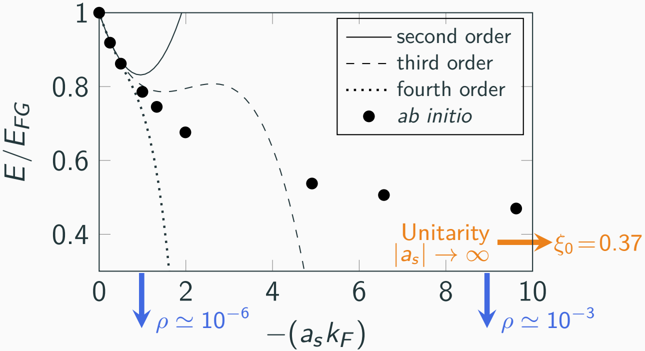

ab initio: [Carlson et al.] MBPT: [Wellenhofer et al. (2018)]

Neutron Matter

$a_s = -18.9$ fm $\qquad r_s = 2.7$ fm

Non-empirical DFT for ultracold atom? $r_s = 0$

Low density limit as a guidance

$$\frac{E}{E_{FG}} = 1 + (a_s k_F)\gamma_1 + (a_s k_F)^2\gamma_2 + \cdots$$

Unitary limit as a guidance

$$\frac{E}{E_{FG}} \xrightarrow[|a_s|\to\infty]{} \xi_0 = 0.37$$ (accepted value)

- non-empirical $\rightarrow f(a_s)$
- DFT $\rightarrow f(\rho)$ or $f(k_F)$
- **finite limit at unitarity**

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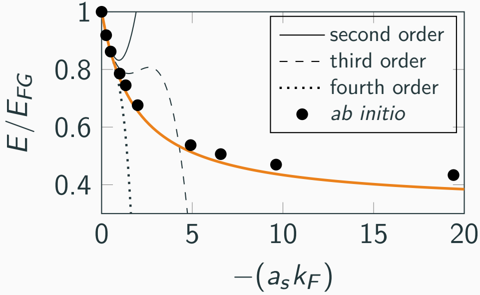

Minimal Padé approximation

$$\frac{E}{E_{FG}} = 1 + \frac{(a_s k_F)\gamma_1}{1 - (a_s k_F)\gamma_2/\gamma_1}$$

✓ valid up to second order in $(a_s k_F)$

✗ incorrect Bertsch parameter ($\simeq 0.32$)

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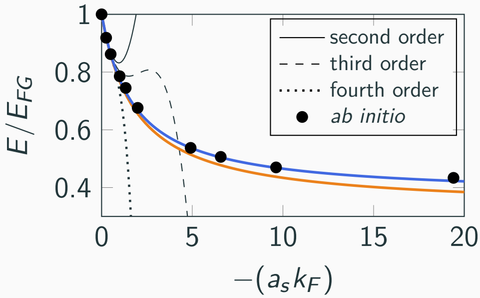

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Padé approximation + constraint

$$\frac{E}{E_{FG}} = 1 + \frac{(a_s k_F)\gamma_1}{1 - (1-\xi_0)^{-1}(a_s k_F)\gamma_1}$$

✗ miss of the second order in $(a_s k_F)$

✓ exact Bertsch parameter

[Lacroix, PRA94 (2016)]

Non-empirical DFT including effective range $r_s \neq 0$

Low density limit as a guidance

$$\frac{E}{E_{FG}} = 1 + (a_s k_F)\gamma_1 + (a_s k_F)^2[\gamma_2 + (r_s k_F)\nu_1] + \cdots$$

[Fetter & Walecka book]

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Unitary limit as a guidance

$$\frac{E}{E_{FG}} \xrightarrow[|a_s|\to\infty]{} \xi_0 + (r_s k_F)\eta_e + (r_s k_F)^2\delta_e + \cdots$$

[Forbes et al., PRA86 (2012)]

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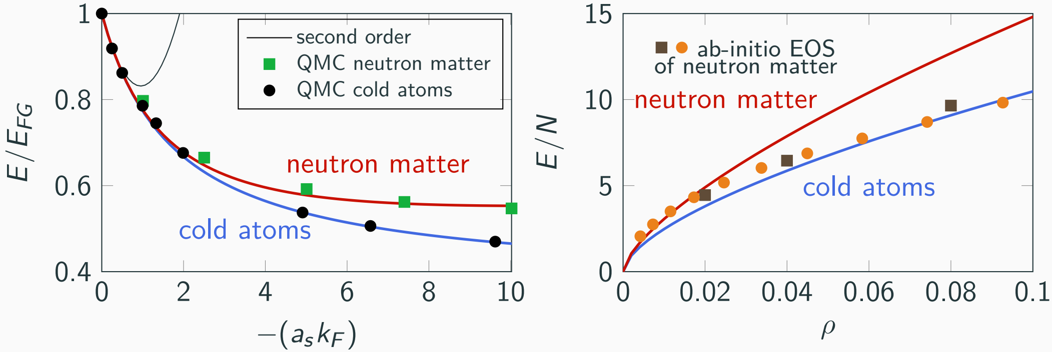
Padé approximation + constraint [Lacroix, AB, et al., PRC 95 (2017)]

$$\frac{E}{E_{FG}} = 1 + \underbrace{\frac{(a_s k_F)\gamma_1}{1 - (1-\xi_0)^{-1}(a_s k_F)\gamma_1}}_{\substack{\text{zero range part} \\ \to \text{ 2 parameters } \gamma_1, \xi_0}} + \underbrace{\frac{(a_s k_F)^2 (r_s k_F)\nu_1 \times \left[1 - (a_s k_F)\sqrt{\nu_1/\eta_e}\right]^{-1}}{1 - (a_s k_F)\sqrt{\nu_1/\eta_e} + (a_s k_F)(r_s k_F)\delta_e/\eta_e}}_{\text{effective range part } \to \text{ 3 parameters } \nu_1, \eta_e, \delta_e}$$

Equation of states of dilute neutron matter



QMC: [Carlson et al.]
ab-initio EOS: [Pandharipande et al.]

Large effective range effect

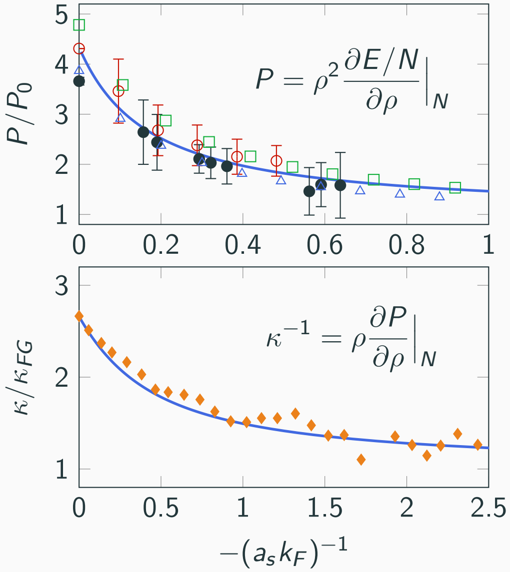
Importance of unitarity → simplicity?

Range of validity

- Lee-Yang: $\rho \lesssim 10^{-6}$ fm^{-3}
- New functional: $\rho \lesssim 10^{-2}$ fm^{-3}

[Lacroix, AB, et al., PRC 95 (2017)]

Thermodynamics of ultracold atoms



Theories

- ○ [Bulgac et al.,PRA78 (2008)]
- □ [Haussmann et al., PRA75 (2007)]
- △ [Hu et al., Europhys. Lett. 74 (2006)]

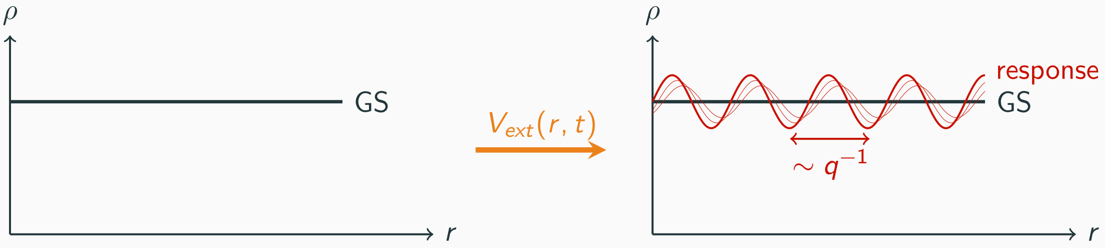
Experiments

- ● [Navon et al., Science 328 (2010)]
- ♦ [Horikoshi et al., PRX7 (2017)]

+ systematic study
of effective range effect ($r_s \neq 0$)

[AB, Lacroix, PRC97 (2018)]

Linear response theory for infinite matter

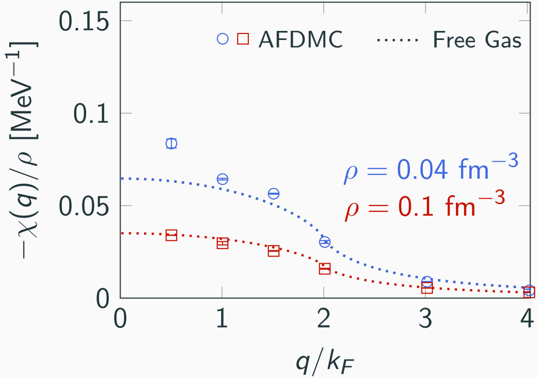


$$V_{ext} = \sum_j \phi(\boldsymbol{q},\omega) e^{i\boldsymbol{q}\cdot\boldsymbol{r}_j - i\omega t} \longmapsto \delta\rho = -\chi(\boldsymbol{q},\omega)\phi(\boldsymbol{q},\omega)$$

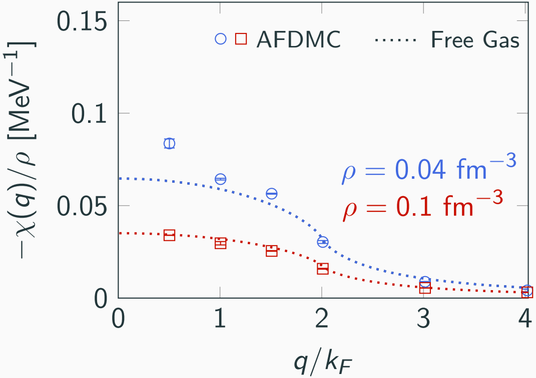
Static response function

$$\chi(q) = \lim_{\omega\to 0} \chi(q,\omega) \qquad \text{(time indep.}\, V_{ext}\text{)}$$

close to the response
of the Free Gas

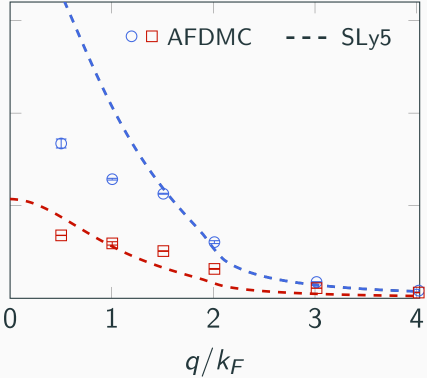


AFDMC: [Buraczynski, Gezerlis, PRC95 (2017)] close to the response of the Free Gas



AFDMC: [Buraczynski, Gezerlis, PRC95 (2017)]

standard empirical functional strongly disagree



SLy5: [Pastore et al., Phys. Rep. 563 (2015)]

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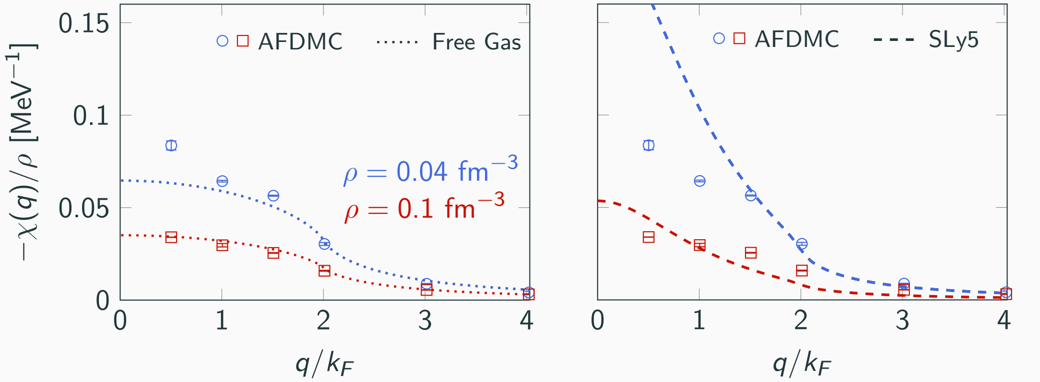

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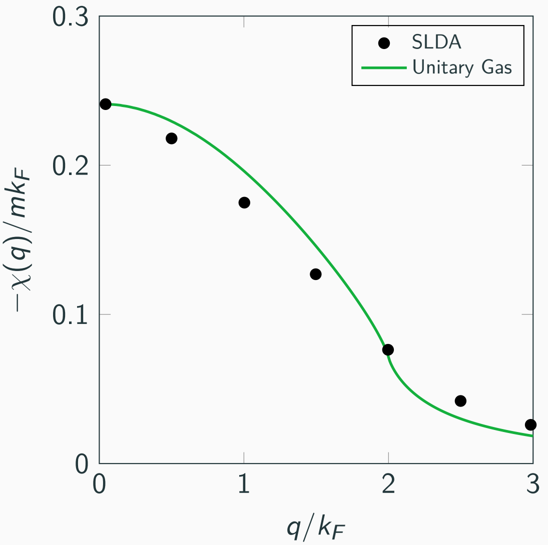
⟼ **Motivate the re-analysis using the new non-empirical functional**



Static response of Unitary Gas

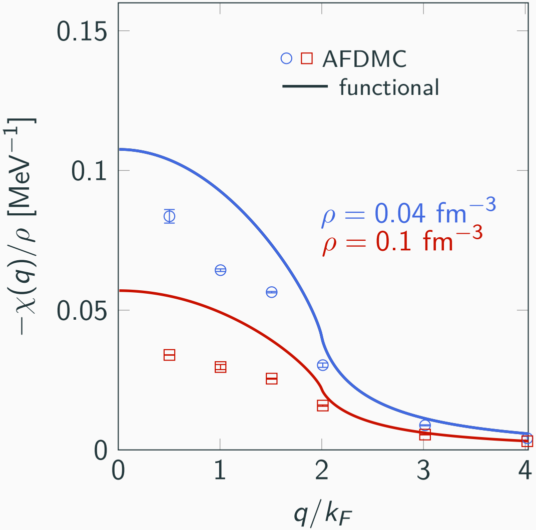
$|a_s| \to \infty$ **and** $r_s = 0$

Superfluid Local Density Approximation (SLDA) $\to$ 3 parameters

[AB, Lacroix, PRC97 (2018)]

SLDA: [Forbes & Sharma, PRA90 (2014)]

Linear response of neutron matter: effective range effect



functional: $E = E(a_s k_F, r_s k_F)$

✓ Better than standard empirical DFT

Static response of neutron matter from the non-empirical DFT

- No effective mass: $m^\star = m$

AFDMC: [Buraczynski, Gezerlis, PRC95 (2017)]
[AB, Lacroix, PRC97 (2018)]

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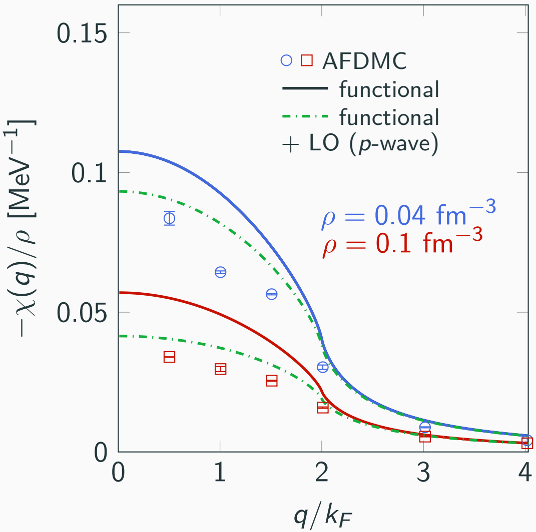


functional: $E = E(a_s k_F, r_s k_F)$

✓ Better than standard empirical DFT

Static response of neutron matter from the non-empirical DFT

- No effective mass: $m^\star = m$
- Adding the leading order of the p-wave into the functional, i.e.: $E \to E + \gamma_p (a_p k_F)^3 E_{FG}$

AFDMC match Free Gas response = compensation effect of many contributions?

AFDMC: [Buraczynski, Gezerlis, PRC95 (2017)]
[AB, Lacroix, PRC97 (2018)]

Summary

- **New non-empirical DFT linked directly to the LECs (a_s,r_s)**
  - ✓ importance of unitary limit
  - ✓ large effective range effect
  - ∼ semi-empirical approach
- **Applications: thermodynamics and linear response**
  - ✓ very promising approach
  - ✗ incomplete description $\longrightarrow$ quasi-particle properties

Second part of the thesis

1. justify the functionals obtained *"intuitively"* starting from a more rigorous non-perturbative many-body theory
2. extend the study to the self-energy to obtain the quasi-particle properties

Basics of diagrammatic framework at zero temperature

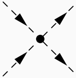
$$E = E_{FG} + E^{(1)} + E^{(2)} + \cdots$$

[Fetter & Walecka book]

$G(\omega, \boldsymbol{k})$  $= \dfrac{n_k}{\omega - e_k + i0^-} + \dfrac{1 - n_k}{\omega - e_k + i0^+}$

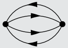
$n_k = \Theta(k_F - k)$: occupation numbers

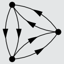
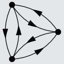
$e_k = k^2/2m$: single particle energy (FG)

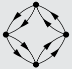
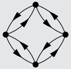
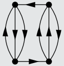
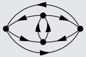
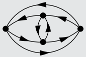
$\langle \boldsymbol{k} | V_{EFT} | \boldsymbol{k'} \rangle$  $= C_0 = 4\pi a_s / m$

Contributing energy diagrams

$E^{(1)} =$  $\rightarrow (a_s k_F) \rightarrow$ *Hartree – Fock*

$E^{(2)} =$  $\rightarrow (a_s k_F)^2 \rightarrow$ *Lee – Yang*

$E^{(3)} =$  $+$ 

$E^{(4)} =$  $+$  $+$  $+$  $+$ 

complexity

Basics of diagrammatic framework at zero temperature

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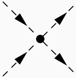
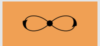
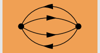
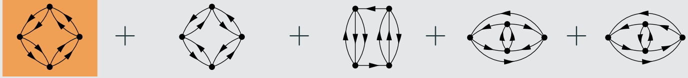
$\langle \boldsymbol{k} | V_{EFT} | \boldsymbol{k}' \rangle$  $= C_0 = 4\pi a_s/m$

Contributing energy diagrams [Ladder approximation]

$E^{(1)} =$  $\rightarrow (a_s k_F) \rightarrow Hartree - Fock$

$E^{(2)} =$  $\rightarrow (a_s k_F)^2 \rightarrow Lee - Yang$

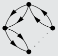
$E^{(3)} =$ 

$E^{(4)} =$ 

complexity

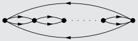
Ladder approximation for the energy

$$E_{int} = \sum_{n=1}^{\infty}$$



$$= \frac{80E_{FG}}{\pi k_F^5} \int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} t dt \, \text{atan} \frac{(a_s k_F)\pi I_*(s,t)}{\pi - (a_s k_F) R(s,t)}$$

$$E_{int}^{pp} = \sum_{n=1}^{\infty}$$



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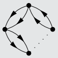
[Kaiser, NPA860 (2011)] (no pairing, no self-consistency)

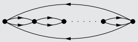
$$F(s,t) = 1 + \frac{s}{k_F} - \frac{t}{k_F} \ln \left| \frac{k_F + s + t}{k_F + s - t} \right| + \frac{k_F^2 - s^2 - t^2}{2 s k_F} \ln \left| \frac{(k_F + s)^2 - t^2}{k_F - s^2 - t^2} \right|$$

$$R(s,t) = F(s,t) + F(-s,t)$$

$$I_*(s,t) = \begin{cases} t/k_F & \text{for } 0 \le t < k_F - s \\ (k_F^2 - s^2 - t^2)/2 s k_F & \text{for } k_F - s \le t < \sqrt{k_F^2 - s^2} \end{cases}$$

Ladder approximation for the energy

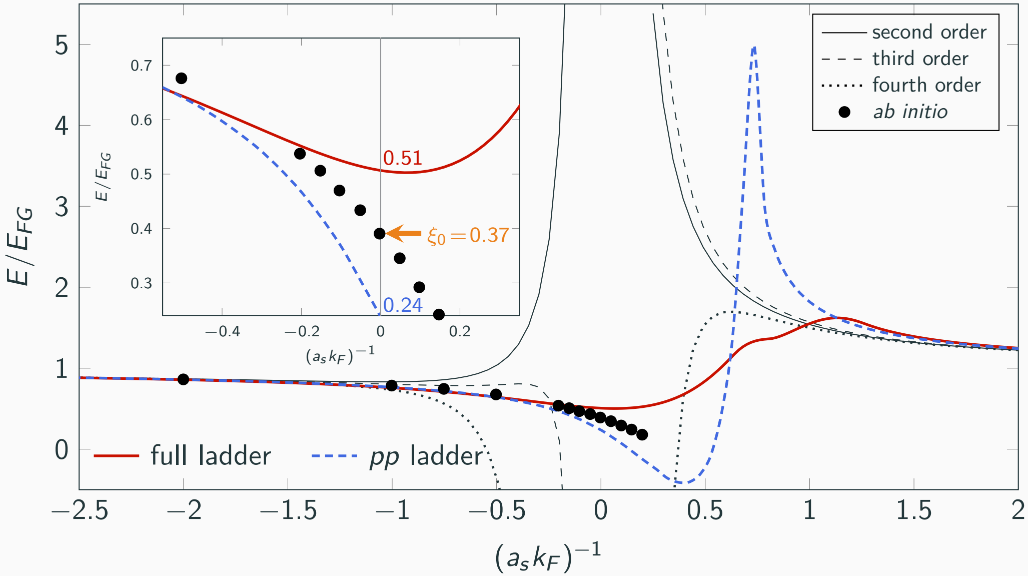
$$E_{int} = \sum_{n=1}^{\infty}$$  $$= \frac{80 E_{FG}}{\pi k_F^5} \int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} t dt \ \text{atan} \frac{(a_s k_F) \pi I_*(s,t)}{\pi - (a_s k_F) R(s,t)}$$

$$E_{int}^{pp} = \sum_{n=1}^{\infty}$$  $$= \frac{80 E_{FG}}{\pi k_F^5} \int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} t dt \ \frac{(a_s k_F) \pi I_*(s,t)}{\pi - (a_s k_F) F(s,t)}$$

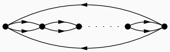
[Kaiser, NPA860 (2011)] (no pairing, no self-consistency)

- ✓ Contain terms to all order in $(a_s k_F)$ in a compact form
- ✓ Expansion in $(a_s k_F) \to$ Lee – Yang formula
- ✓ Finite limit at unitarity ($|a_s| \to \infty$)
- ✗ Implicit function of $\rho = k_F^3 / 3\pi^2$ (goal: explicit function)

Ladder approximation for the energy II



Phase-space Approximation



$$\frac{E_{pp}}{E_{FG}} = 1 + \frac{80}{\pi k_F^5} \underbrace{\int s^2 ds \int t dt}_{phase\ space} \frac{(a_s k_F)\pi I(s,t)}{1 - (a_s k_F/\pi) F(s,t)} \xrightarrow[|a_s k_F| \to \infty]{} 0.24$$

Phase-space Approximation of *pp* ladder resummation

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi} \frac{(a_s k_F)}{1 - (a_s k_F/\pi)\langle F \rangle} \xrightarrow[|a_s k_F| \to \infty]{} 0.32$$

[Heiselberg, PRA63 (2001)] [Schäfer et al., NPA762 (2005)] [Haussmann et al., PRA75 (2007)]

✓ Match the Lee – Yang expansion at second order

$\langle F \rangle = \frac{6}{35}(11 - 2\ln 2)$

∼ More predictive near unitarity

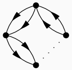
$\xi_0 \simeq 0.37$ (accepted value)

Adjust eventually $\langle F \rangle$ on unitary limit

✓ Exact at unitarity $|a_s| \to \infty$

✗ Lee – Yang expansion

Phase-space Approximation II



$$\frac{E}{E_{FG}} = 1 + \frac{80}{\pi k_F^5} \underbrace{\int s^2 ds \int tdt}_{phase\ space} \operatorname{atan} \frac{(a_s k_F) I(s,t)}{1 - (a_s k_F/\pi) R(s,t)} \underset{|a_s k_F| \to \infty}{=} 0.51$$

Phase-space Approximation of full ladder resummation

$$\frac{E}{E_{FG}} = 1 + \frac{16}{3\pi} \operatorname{atan} \frac{5/24 (a_s k_F)}{1 - (a_s k_F/\pi)\langle R \rangle} \underset{|a_s k_F| \to \infty}{=} \mathbf{0.36}$$

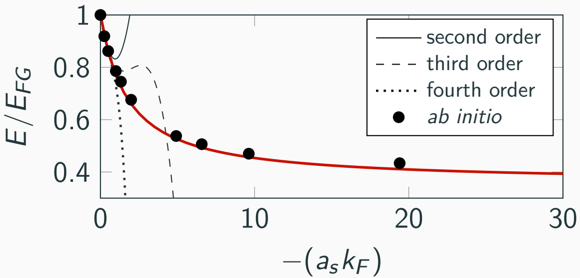
- ✓ Unitary limit well reproduced

  $\xi_0 \simeq 0.37$ (accepted value)

- ✓ Match the Lee – Yang expansion at second order

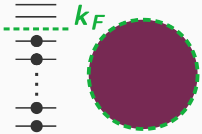
  $\langle R \rangle = \frac{6}{35}(11 - 2 \ln 2)$

[AB, Lacroix, J. Phys. G **46**, (2019)]



Test particle method

$$E_{int} = \sum_{kk'} V_{eff}(k, k') n_k n_{k'}$$



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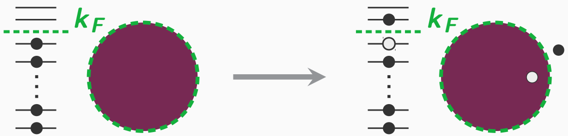
Low-lying excited states $\quad n_k \to n_k + \delta n_k$

$$\delta E = \sum_k \Sigma^\star(k) \delta n_k \longmapsto$$

$$\Sigma^\star(k) = U(k) + iW(k) = \frac{\delta E}{\delta n_k}$$



$$\epsilon_k = \frac{k^2}{2m} + U(k) \quad \text{(single-particle energy)}$$

$$\frac{1}{2\gamma_k} = -W(k) \quad \text{(life-time)}$$

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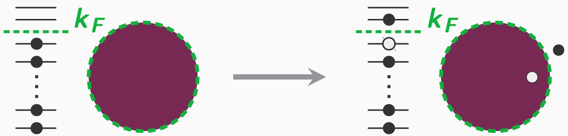
Close to Fermi surface

$$v_{k_F} \equiv \partial_k \epsilon_k \big|_{k=k_F} \equiv k_F / m^\star$$

$$\epsilon_k = \epsilon_{k_F} + (k - k_F)\frac{k_F}{m^\star} + \cdots$$



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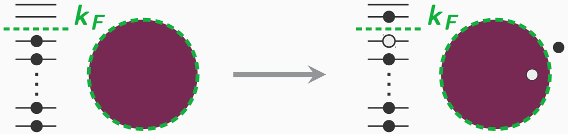

$$\epsilon_k = \frac{k^2}{2m} + U(k) \quad \text{(single-particle energy)}$$

$$\frac{1}{2\gamma_k} = -W(k) \quad \text{(life-time)}$$

Hugenholtz – van Hove theorem (HvH)

$$\mu = E(N+1) - E(N) = \frac{\partial E}{\partial N} = \epsilon_{k_F}$$

[Hugenholtz, Van Hove, Physica XXIV (1958)]

$$E_{int} = E_{(1)} + E_{(2)} + \cdots$$

$$E_{(1)} = \frac{10}{9\pi}(a_s k_F) E_{FG}$$

$$E_{(2)} = E_{FG}\frac{4}{21\pi^2}(11 - 2\ln 2)(a_s k_F)^2$$

Illustration with the second order

$$E_{int} = E_{(1)} + E_{(2)} + \cdots$$

$$E_{(1)} = \frac{10}{9\pi}(a_s k_F) E_{FG}$$

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↓

$$\Sigma^\star(k) = \Sigma^\star_{(1)}(k) + \Sigma^\star_{(2)}(k) + \cdots$$

$$\Sigma^\star_{(1)}(k) = \frac{4}{3\pi}(a_s k_F)\mu_{FG}$$

$$\Sigma^\star_{(2)}(k) = \mu_{FG}\left[\phi_2(k) + i\chi_2(k)\right](a_s k_F)^2$$

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$$\phi_2(k) \underset{k\sim k_F}{=} \frac{4}{15\pi^2}(11 - 2\ln 2) + 2\left(\frac{k}{k_F} - 1\right)\frac{8}{15\pi^2}(1 - 7\ln 2) + \cdots$$

$$\epsilon(k) = \frac{k^2}{2m} + \overbrace{Re[\Sigma^\star(k)]}^{U(k)}$$

$$\underset{k\sim k_F}{\equiv} \boldsymbol{\mu} + (k - k_F)\frac{k_F}{\boldsymbol{m^\star}} + \cdots$$

$$E_{int} = E_{(1)} + E_{(2)} + \cdots$$

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$$\underset{k\sim k_F}{\equiv} \boldsymbol{\mu} + (k - k_F)\frac{k_F}{\boldsymbol{m^\star}} + \cdots$$

$$\longmapsto \begin{cases} \dfrac{\mu}{\mu_{FG}} = 1 + \dfrac{4}{3\pi}(a_s k_F) + \dfrac{4}{15\pi^2}(11 - 2\ln 2)(a_s k_F)^2 + \cdots \\ \dfrac{m}{m^\star} = 1 + \dfrac{8}{15\pi^2}(1 - 7\ln 2)(a_s k_F)^2 + \cdots \end{cases}$$

Ladder approximation: analytical results

$$\Sigma^\star(k) = U(k) + iW(k) \qquad U(k < k_F) = \frac{8}{m\pi^2}\int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} t dt\, \mathcal{U}(s, t, k < k_F)$$

[Kaiser, EPJA49 (2013)]

✓ valid at low density (Galitskii formula)

✓ finite limit at unitarity $|a_s k_F| \to \infty$

Ladder approximation: analytical results

$$\Sigma^\star(k) = U(k) + iW(k) \qquad U(k<k_F) = \frac{8}{m\pi^2}\int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} tdt\, \mathcal{U}(s,t,k<k_F)$$

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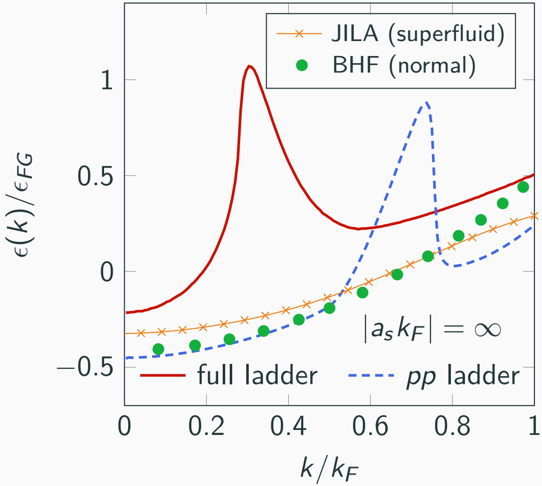


- ✓ valid at low density (Galitskii formula)
- ✓ finite limit at unitarity $|a_s k_F| \to \infty$
- ✗ bad predictivity power for $|a_s k_F| \gg 1$
- ✗ strong dependence of retained diagrams (cf. energy)

BHF: [Doggen & Kinnumen (2015)]
JILA exp.: [Stewart et al., Nature **454** (2008)]

Strategy for the self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s, t)$$

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s, t, k)$$

Strategy for the self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s,t)$$

$$\Uparrow \Downarrow$$

$$\mu = \left.\frac{\partial E}{\partial N}\right|_V = \epsilon(k_F)$$

$\Uparrow$ $k = k_F$ HvH theorem

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s,t,k)$$

Strategy for the self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s,t)$$

PS approx.

Lee − Yang

$$E_{PS}$$

$$\mu = \left.\frac{\partial E}{\partial N}\right|_V = \epsilon(k_F)$$

$k = k_F$

HvH theorem

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s,t,k)$$

Strategy for the self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s,t)$$

PS approx. — **Lee – Yang** → E_{PS}

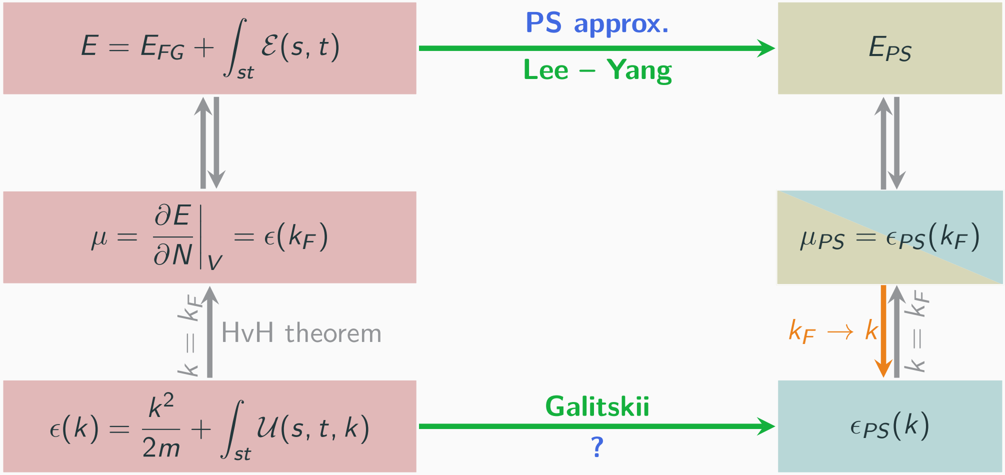
$$\mu = \left.\frac{\partial E}{\partial N}\right|_V = \epsilon(k_F)$$

$k = k_F$ — HvH theorem

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s,t,k)$$

Galitskii — **?** → $\epsilon_{PS}(k)$

Strategy for the self-energy resummation



Partial phase-space average approximation

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi} \frac{(a_s k_F)}{1 - (a_s k_F/\pi)\frac{9\pi^2}{14}\phi_2(k_F)}$$

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$$\phi_2(k_F) \to \phi_2(k)$$

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✓ Lee-Yang Formula

$$\frac{\mu}{\mu_{FG}} = 1 + \frac{4}{3}\frac{(a_s k_F)}{\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k_F)} + \frac{2}{9}\frac{(a_s k_F)^2 \frac{9\pi^2}{14}\phi_2(k_F)}{\left[\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k_F)\right]^2}$$

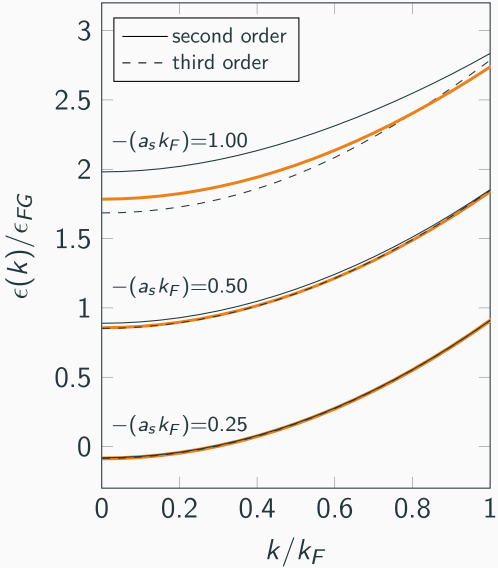
$$\phi_2(k_F) \to \phi_2(k)$$

✓ HvH theorem $\mu = \epsilon(k_F)$

$$\frac{\epsilon(k)}{\epsilon_{FG}} = \frac{k^2}{k_F^2} + \frac{4}{3}\frac{(a_s k_F)}{\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k)} + \frac{2}{9}\frac{(a_s k_F)^2 \frac{9\pi^2}{14}\phi_2(k)}{\left[\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k)\right]^2}$$

✓ Galitskii Formula

Results



- ✓ exact expansion up to $(a_s k_F)^2$
- ✓ simpler function of the density

MBPT: [Platter et al., NPA714 (2003)]
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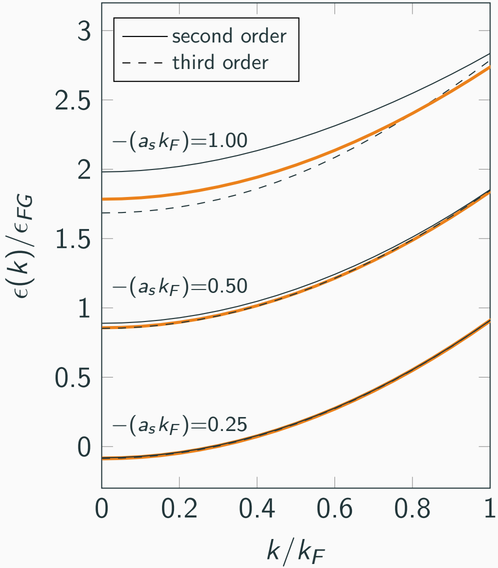
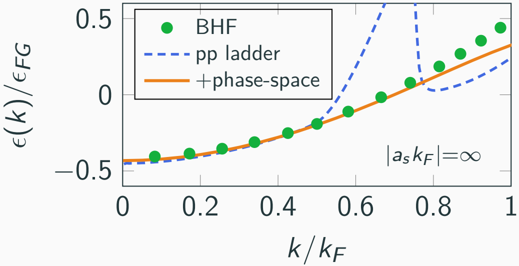
Results





- ✓ exact expansion up to $(a_s k_F)^2$
- ✓ simpler function of the density
- ✓ pathologies removed for $|a_s k_F| \gg 1$

MBPT: [Platter et al., NPA714 (2003)]
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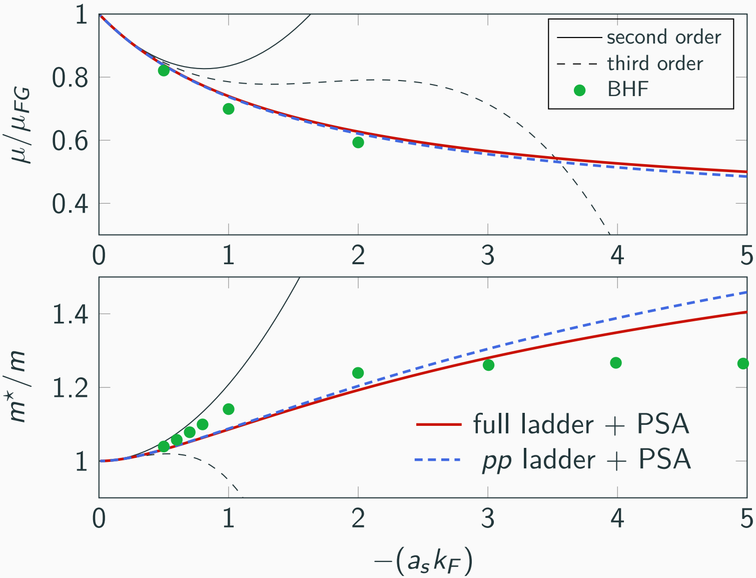
Quasi-particle properties



$$\mu = \epsilon(k_F)$$

BHF: [Doggen & Kinnumen (2015)]
MBPT: [Platter et al., NPA714 (2003)]

$$\frac{m}{m^\star} = \frac{m}{k_F} \frac{\partial \epsilon_k}{\partial k}\bigg|_{k_F}$$

- ✓ Galitskii formula
- ✓ finite limit at unitarity
- ✓ beyond the perturbative regime

[AB, Lacroix, J. Phys. G **46**, (2019)]

Summary

- ⊙ **non-empirical DFT**
  - ✓ **Study of the DFT as a semi-empirical function of the LECs (a_s,r_s)**
  - ✓ **Applications to cold atoms & neutron matter**
    (equation of state & thermodynamics + static response)
- ⊙ **Non-perturbative resummation technique**
  - ✓ **Study at energy level → Phase-Space Approximation**
  - ✓ **link with the semi-empirical DFT = justification**
- ⊙ **Study of the self-energy**
  - ✓ **generalization of the Phase-Space Approximation to the self-energy**
  - ✓ **quasi-particle properties in the non-perturbative regime**

Outlooks and perspectives

Perspectives and discussions towards non-empirical DFT

- Analytical developments with simple interactions
  - ? more realistic interaction (p-wave, ...) ? superfluidity
- Cross-fertilization: DFT vs *ab initio* [Grasso, Prog. in Part. and Nucl. Phys. **106** (2019)]

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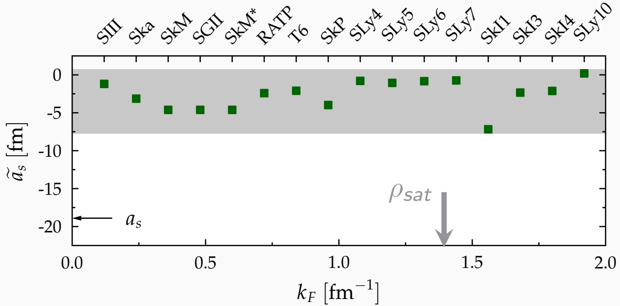
✓ **Link with the standard DFT (renormalization of the LECs)**



Skyrme functionals:
$\widetilde{a}_s(k_F) = \widetilde{a}_s = mt_0(1 - x_0)/4\pi$

[Lacroix, AB, et al., PRC 95 (2017)]
[Furnstahl, *EFT for DFT* (2008)]

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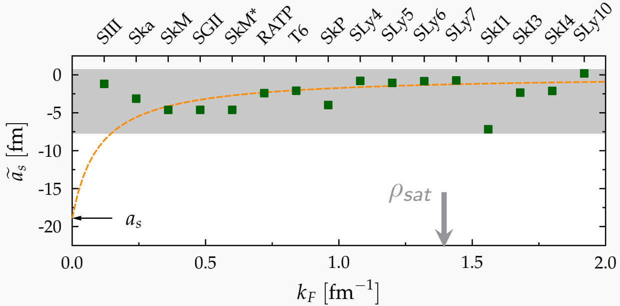

DFT re-written as:

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi}\left[\tilde{a}_s(k_F)k_F\right]$$

Skyrme functionals:
$\tilde{a}_s(k_F) = \tilde{a}_s = mt_0(1-x_0)/4\pi$

[Lacroix, AB, et al., PRC 95 (2017)]
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