Quasi-particle properties of Fermi gas from low density to unitary limit

Bridging nuclear *ab-initio* and density functional theories

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Theory group seminar – IPNO
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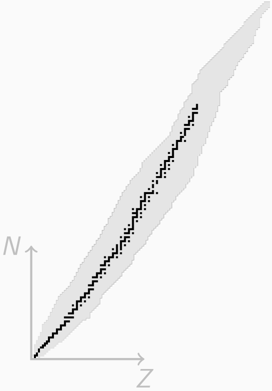
Context and motivation



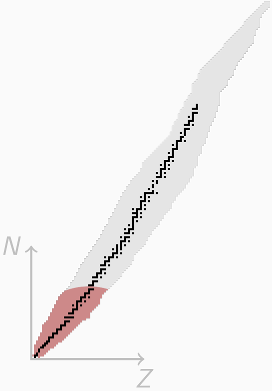
Context and motivation

$V_2, V_3, \ldots$ $\rightarrow$ LECs

Ab-initio
BHF,SCGF,QMC,...

static (also a bit of dynamic) properties of some nuclei and infinite nuclear matter

Results



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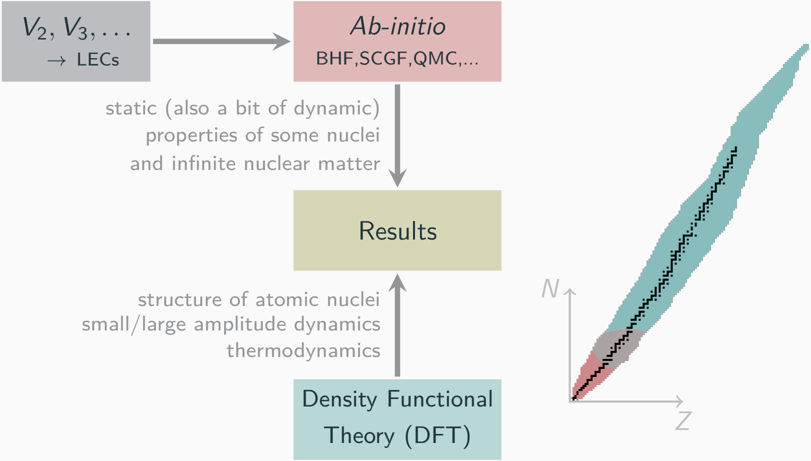

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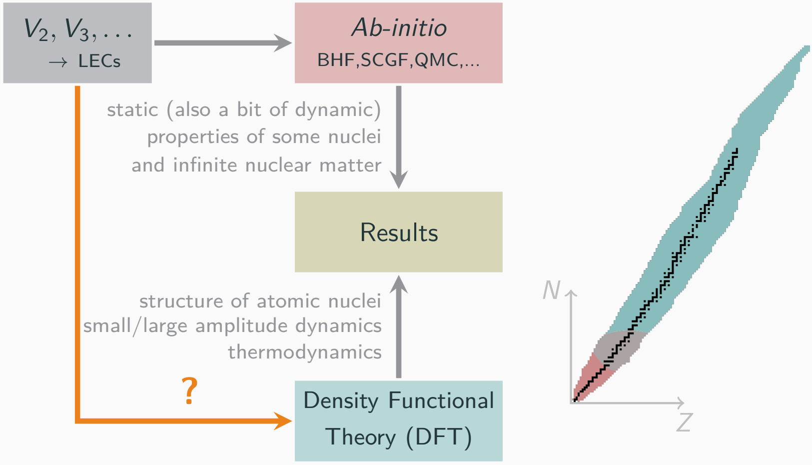


How to relate the bare interaction to DFT and make it less empirical?

In this work $\longrightarrow$ a focus on infinite matter

Table of contents

The low-density Fermi gas limit: EFT guidance

$$\left\langle \boldsymbol{k} \middle| V_{EFT} \middle| \boldsymbol{k'} \right\rangle = \underbrace{C_0 + \frac{C_2}{2}\left(\boldsymbol{k}^2 + \boldsymbol{k'}^2\right)}_{s-wave} + \cdots$$

[Steele and Furnstahl, NPA762 (2000)]
[Beane *et al.*, nucl-th/0008064 (2000)]
[Hammer and Furnstahl, NPA678 (2000)]

$$C_0 = \frac{4\pi}{m} a_s \qquad C_2 = \frac{2\pi}{m} a_s^2 r_s$$

Neutron Matter

$$a_s = -18.9 \text{ fm}$$

$$r_s = 2.7 \text{ fm}$$

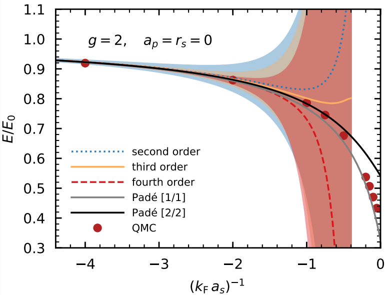
Constructive MBPT

✓ GS energy up to fourth order

[Wellenhofer *et al.*, arXiv (2019)]

UV divergence properly treated

[Kaplan, Savage, Wise, NPB534 (1998)]



Lee-Yang energy density functional

$$E(\rho) = E_{FG} + E^{(1)} + E^{(2)} + \cdots \qquad \left[E_{FG} = \frac{3}{5}\frac{k_F^2}{2m}\rho \;\middle|\; \rho = \frac{k_F^3}{3\pi^2} \right]$$

$$= E_{FG}\left[1 + \frac{10}{9\pi}(a_s k_F) + \frac{4}{21\pi^2}(11 - 2\ln 2)(a_s k_F)^2 + \cdots\right]$$

$$= \frac{3(3\pi^2)^{2/3}}{10m}\rho^{5/3} + \frac{\pi a_s}{m}\rho^2 + \frac{6(11-2\ln 2)a_s^2}{35(3\pi^2)^{-1/3}m}\rho^{7/3} + \cdots$$

✓ analytic dependence in term of ρ and a_s

Difficulties of the perturbative approach

- Perturbative approach valid if $|a_s k_F| \ll 1$

  Neutron matter: $a_s = -18.9$ fm $\rightarrow \rho \lesssim 10^{-6}$ fm$^{-3} \ll \rho_0 \simeq 0.16$ fm^3
- Non perturbative approaches
  - Standard MB techniques: BHF, SCGF, QMC, AFDMC, ...
    - ✗ non-analytic in $a_s k_F$
  - Resummation technique
    - ✓ analytic in $a_s k_F$ (compatible with a DFT point of view)

Lee-Yang energy density functional

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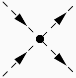
Basics of diagrammatic framework at zero temperature

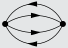
[Hammer and Furnstahl, NPA678 (2000)]

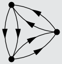
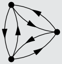
$G(\omega, \boldsymbol{k}) =$  $= \dfrac{n_k}{\omega - e_k + i0^-} + \dfrac{1 - n_k}{\omega - e_k + i0^+}$

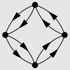
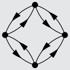
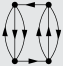
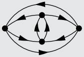
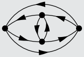
$\langle \boldsymbol{k} | V_{EFT} | \boldsymbol{k'} \rangle =$  $= C_0$ $\quad [n_k = \Theta(k_F - k) \mid e_k = k^2/2m]$

Contributing energy diagrams

$E_{(1)} =$  $\rightarrow (a_s k_F) \rightarrow$ *Hartree − Fock*

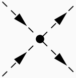
$E_{(2)} =$  $\rightarrow (a_s k_F)^2 \rightarrow$ *Lee − Yang*

$E_{(3)} =$  $+$ 

$E_{(4)} =$  $+$  $+$  $+$  $+$ 

[Hammer and Furnstahl, NPA678 (2000)]

$G(\omega, \boldsymbol{k}) =$  $= \dfrac{n_k}{\omega - e_k + i0^-} + \dfrac{1 - n_k}{\omega - e_k + i0^+}$

$\langle \boldsymbol{k} | V_{EFT} | \boldsymbol{k'} \rangle =$  $= C_0$ $\quad [n_k = \Theta(k_F - k) \mid e_k = k^2/2m]$

Contributing energy diagrams [Ladder approximation]

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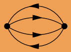
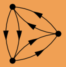
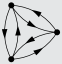
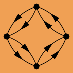
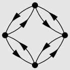
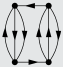
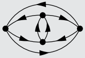
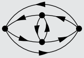
$E_{(2)} =$  $\rightarrow (a_s k_F)^2 \rightarrow$ *Lee – Yang*

$E_{(3)} =$  $+$ 

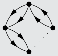
$E_{(4)} =$  $+$  $+$  $+$  $+$ 

[Kaiser, NPA **860** (2011)]

Ladder approximation for the energy

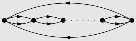
Energy resummation

$$E_{int} = \sum_{n=1}^{\infty}$$



$$= \frac{80}{\pi k_F^5} E_{FG} \int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} t dt \ \mathrm{atan} \frac{(a_s k_F)\pi I(s,t)}{\pi - (a_s k_F) R(s,t)}$$

$$E_{int}^{pp} = \sum_{n=1}^{\infty}$$



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[Kaiser, NPA **860** (2011)] (no pairing, no self-consistency)

$$F(s,t) = 1 + \frac{s}{k_F} - \frac{t}{k_F} \ln \left| \frac{k_F + s + t}{k_F + s - t} \right| + \frac{k_F^2 - s^2 - t^2}{2 s k_F} \ln \left| \frac{(k_F + s)^2 - t^2}{k_F - s^2 - t^2} \right|$$

$$R(s,t) = F(s,t) + F(-s,t)$$

$$I(s,t) = \begin{cases} t/k_F & \text{for } 0 \le t < k_F - s \\ (k_F^2 - s^2 - t^2)/2 s k_F & \text{for } k_F - s \le t < \sqrt{k_F^2 - s^2} \end{cases}$$

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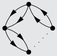

$$= \frac{80}{\pi k_F^5} E_{FG} \int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} t dt \; \mathrm{atan} \frac{(a_s k_F) \pi I(s,t)}{\pi - (a_s k_F) R(s,t)}$$

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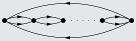

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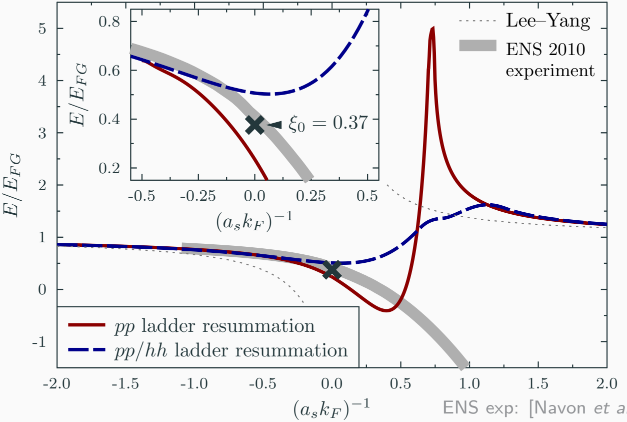
[Kaiser, NPA **860** (2011)] (no pairing, no self-consistency)

- ✓ Contains terms to all order in $(a_s k_F)$ in a compact form
- ✓ Expansion in $(a_s k_F) \to$ Lee–Yang formula
- ✓ Finite limit at unitarity ($a_s \to \infty$)
- ✗ Implicit function of ρ (goal: explicit function)

Ladder approximation for the energy



ENS exp: [Navon *et al.*, Science **328** (2010)]

- ✓ correct limit at $a_s k_F \ll 1$ (Lee-Yang expansion)
- ✓ finite limit at unitarity
- ✗ strong dependence of retained diagrams
- ✗ complicated function of $(a_s k_F)$

Phase-space average Approximation (PSA)

$$\frac{E}{E_{FG}} = 1 + \frac{80}{\pi k_F^5} \int s^2 ds \int t dt \frac{(a_s k_F)\pi I(s,t)}{1 - (a_s k_F/\pi) F(s,t)} \underset{a_s k_F \to \infty}{\longrightarrow} 0.24$$

PSA of *pp* ladder resummation = GPS functional

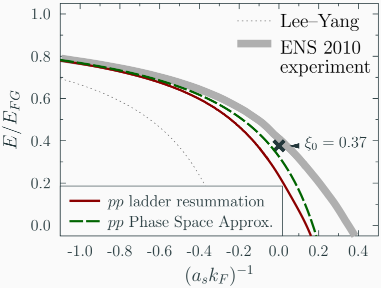
$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi} \frac{(a_s k_F)}{1 - (a_s k_F/\pi)\langle F\rangle} \underset{a_s k_F \to \infty}{\longrightarrow} 0.32$$

[Heiselberg, PRA63 (2001)] [Schäfer *et al.*, NPA762 (2005)] [Haussmann *et al.*, PRA75 (2007)]

✓ Lee–Yang formula

$$\langle F\rangle = \frac{6}{35}(11 - 2\ln 2)$$

∼ More predictive near unitarity: $\xi_0 = 0.37$ (accepted value)



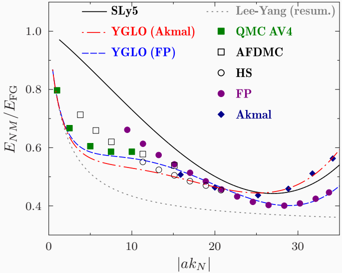
$$\frac{E}{E_{FG}} = 1 + \frac{\frac{10}{9\pi}(a_s k_F)}{1 - \frac{6}{35\pi}(11 - 2\ln 2)(a_s k_F)} \rightarrow \frac{E}{A} = K + \frac{B\rho}{1 - R\rho^{1/3} + C\rho^{2/3}} + D\rho^{5/3} + F\rho^{\alpha+1}$$

YGLO functional

B, R: Lee–Yang (low density)
$\rightarrow$ non-empirical

C, D, F: higher correlations (fit)
$\rightarrow$ empirical

[Yang, Grasso, Lacroix, PRC **94** (2016)]



$$\frac{E}{E_{FG}} = 1 + \frac{\frac{10}{9\pi}(a_s k_F)}{1 - \frac{6}{35\pi}(11 - 2\ln 2)(a_s k_F)} \to 1 + \frac{\frac{10}{9\pi}(a_s k_F)}{1 - \frac{10}{9\pi}(1-\xi_0)^{-1}(a_s k_F)}$$

$$\to \frac{E}{A} = K + \frac{B\rho}{1 - R\rho^{1/3} + C\rho^{2/3}} + D\rho^{5/3} + F\rho^{\alpha+1}$$

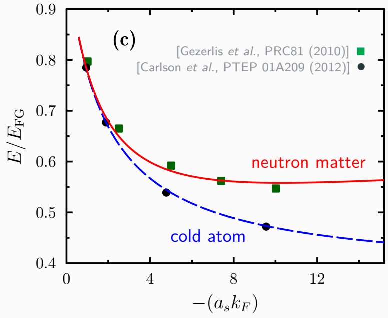
[Lacroix, AB, *et al.*, PRC **95** (2017)]

✓ unitary limit reproduced
✗ Lee-Yang formula

YGLO functional

B, R: Lee–Yang (low density)
→ non-empirical

C, D, F: higher correlations (fit)
→ empirical



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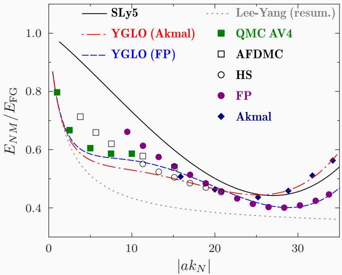


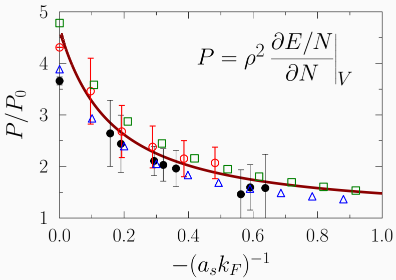
Phase-space average Approximation (PSA)

$$\frac{E}{E_{FG}} = 1 + \frac{80}{\pi k_F^5} \int s^2 ds \int t dt \operatorname{atan} \frac{(a_s k_F) I(s,t)}{1 - (a_s k_F/\pi) R(s,t)} \underset{a_s k_F \to \infty}{=} 0.51$$

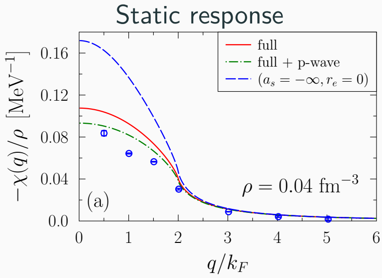
PSA of full ladder resummation = APS functional

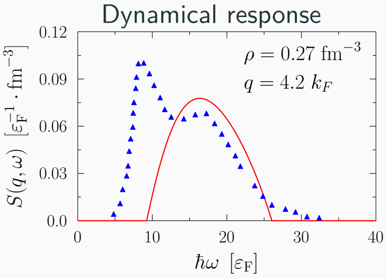
$$\frac{E}{E_{FG}} = 1 + \frac{16}{3\pi} \operatorname{atan} \frac{5/24(a_s k_F)}{1 - (a_s k_F/\pi)\langle R \rangle} \underset{a_s k_F \to \infty}{=} \mathbf{0.36}$$

- ✓ Unitary limit well reproduced (accepted value: $\xi_0 = 0.37$)
- ✓ Exact Lee–Yang expansion
- ✓ **No adjustment !**



[AB, Lacroix, in preparation]





✗ effective mass $m^\star$

✗ pairing gap Δ

[AB, Lacroix, PRC **97** (2018)]

Goal: extend to self-energy → quasi-particle properties (focus on $m^\star$)

Quasi-particle properties: Self-Energy Resummation

Illustration: low density limit

Ladder Resummation

Phase-Space average Approximation

Link with Landau theory of Fermi liquid

$$E_{int} \;=\; \sum_{kk'} V_{eff}(k,k') n_k n_{k'}$$

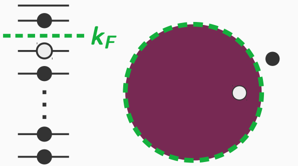
Low-lying excited states $\quad n_k \to n_k + \delta n_k$

$$\delta E \;=\; \sum_k \Sigma^\star(k)\delta n_k \;\longmapsto$$

$$\Sigma^\star(k) \;=\; U(k) + iW(k) \;=\; \frac{\delta E}{\delta n_k}$$

Close to Fermi surface $\quad v_{k_F} \equiv \partial_k \epsilon_k \big|_{k=k_F} \equiv \frac{k_F}{m^\star}$

$$\epsilon_k \;=\; \epsilon_{k_F} + (k - k_F)\frac{k_F}{m^\star} + \cdots$$



[Landau (1957)]

$$\left[\begin{array}{ccc} \epsilon_k & = & \frac{k^2}{2m} + U(k) \\ \frac{1}{2\gamma_k} & = & -W(k) \end{array}\right]$$

Link with Landau theory of Fermi liquid

$$E_{int} = \sum_{kk'} V_{eff}(k,k') n_k n_{k'}$$

Low-lying excited states — $n_k \to n_k + \delta n_k$

$$\delta E = \sum_k \Sigma^\star(k)\delta n_k \longmapsto$$
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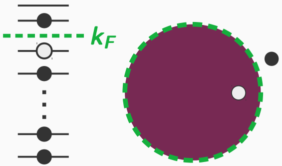
Close to Fermi surface — $v_{k_F} \equiv \partial_k \epsilon_k \big|_{k=k_F} \equiv \frac{k_F}{m^\star}$

$$\epsilon_k = \epsilon_{k_F} + (k - k_F)\frac{k_F}{m^\star} + \cdots$$



[Landau (1957)]

$$\left[\begin{array}{ccc} \epsilon_k & = & \frac{k^2}{2m} + U(k) \\ \frac{1}{2\gamma_k} & = & -W(k) \end{array} \right]$$

Hugenholtz – van Hove theorem (HvH theorem)

$$\mu = E(N+1) - E(N) = \frac{\partial E}{\partial N} = \epsilon_{k_F}$$

[Hugenholtz, Van Hove, Physica **XXIV** (1958)]

Illustration with the second order: Galitskii formula

$$\Sigma^{\star}(k) = \Sigma^{\star}_{(1)}(k) + \Sigma^{\star}_{(2)}(k) + \cdots$$

$$\Sigma^{\star}_{(1)}(k) = \frac{4}{3\pi}(a_s k_F)\mu_{FG} \qquad \Sigma^{\star}_{(2)}(k) = \mu_{FG}\left[\phi_2(k) + i\chi_2(k)\right](a_s k_F)^2$$

e.g. $\phi_2(k) \underset{k\sim k_F}{=} \frac{4}{15\pi^2}(11 - 2\ln 2) + 2\left(\frac{k}{k_F} - 1\right)\frac{8}{15\pi^2}(1 - 7\ln 2) + \cdots$

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Quasi-particle properties

$$\epsilon(k) = \frac{k^2}{2m} + \underbrace{Re[\Sigma^\star(k)]}_{U(k)} \underset{k\sim k_F}{\equiv} \boldsymbol{\mu} + (k - k_F)\frac{k_F}{\boldsymbol{m^\star}} + \cdots$$

$$\longmapsto \begin{cases} \dfrac{\mu}{\mu_{FG}} = 1 + \dfrac{4}{3\pi}(a_s k_F) + \dfrac{4}{15\pi^2}(11 - 2\ln 2)(a_s k_F)^2 + \cdots \\ \dfrac{m}{m^\star} = 1 + \dfrac{8}{15\pi^2}(1 - 7\ln 2)(a_s k_F)^2 + \cdots \end{cases}$$

Ladder approximation: semi-analytical results

$$\Sigma^\star(k) = U(k) + iW(k)$$

$$U(k < k_F) = \frac{8}{m\pi^2}\int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2 - s^2}} tdt\, \mathcal{U}(s, t, k < k_F)$$

Only pp ladder resummation [Kaiser, EPJA 49 (2013)]

$$\mathcal{U}_{pp}(s, t, k < k_F) = \frac{\pi^2}{k_F^3}\left\{\frac{(a_s k_F)^2 \widehat{F}(s, t, k) I_*(s, t)}{\left[\pi - (a_s k_F) F(s, t)\right]^2} - \frac{(a_s k_F)\widehat{I_*}(s, t, k)}{\pi - (a_s k_F) F(s, t)}\right\}$$

Ladder approximation: semi-analytical results

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Only pp ladder resummation [Kaiser, EPJA 49 (2013)]

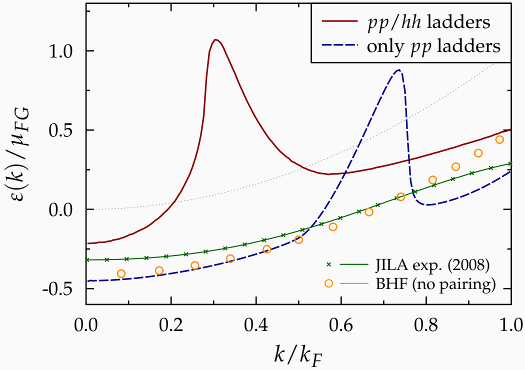
$$\mathcal{U}_{pp}(s,t,k<k_F) = \frac{\pi^2}{k_F^3}\left\{\frac{(a_s k_F)^2 \widehat{F}(s,t,k) I_*(s,t)}{[\pi-(a_s k_F)F(s,t)]^2} - \frac{(a_s k_F)\widehat{I_*}(s,t,k)}{\pi-(a_s k_F)F(s,t)}\right\}$$

Same structure as for energy resummed:

$$E = \frac{8}{m\pi^2}\int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2-s^2}} tdt\, \mathcal{E}(s,t)$$

- ✓ expansion up to second order → Galitskii Formula
- ✓ finite limit at unitarity ($a_s k_F \to \infty$)
- ✗ Implicit function of ρ (goal: explicit function)

Ladder approximation: single-particle energy



- ✓ valid at low density → Galitskii formula
- ✓ finite limit at unitarity ($a_s k_F \to \infty$)
- ✗ non-predictive for $a_s k_F \gg 1$: pathologies
- ✗ strong dependence of retained diagrams

BHF: [Doggen and Kinnumen, Nature (2015)]
JILA experiment: [Stewart, Gaebler, Jin, Nature **454** (2008)]

Phase-Space Average approximation of the resummed self-energy

Focus on the single particle potential $U(k)$ inside the Fermi surface $(k \leq k_F)$

Strategy of the Self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s, t)$$

Strategy of the Self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s, t)$$

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s, t, k)$$

Strategy of the Self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s,t)$$

$$\mu = \left.\frac{\partial E}{\partial N}\right|_V = \epsilon(k_F)$$

$k = k_F$

HvH theorem

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s,t,k)$$

Strategy of the Self-energy resummation

$$E = E_{FG} + \int_{st} \mathcal{E}(s,t)$$

PS approx.

Lee-Yang

$$E_{PS}$$

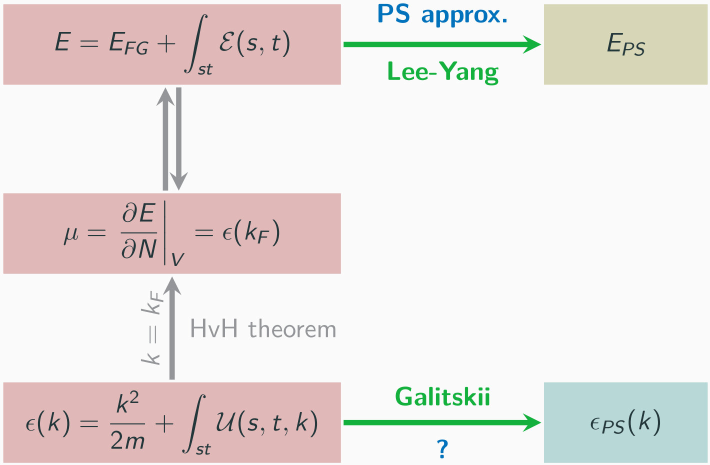
$$\mu = \left.\frac{\partial E}{\partial N}\right|_V = \epsilon(k_F)$$

$k = k_F$

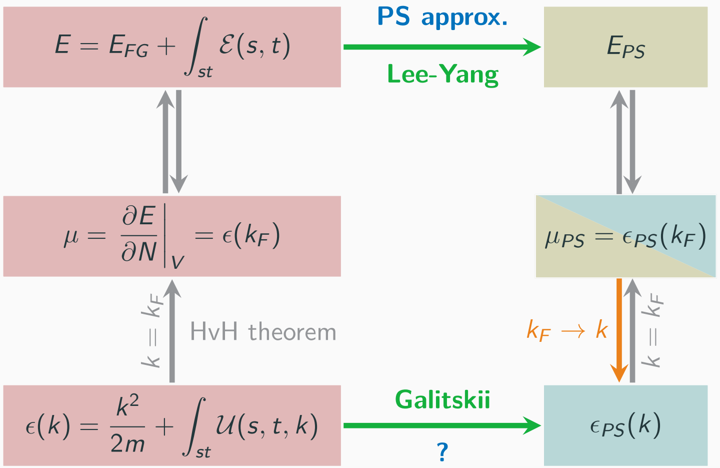
HvH theorem

$$\epsilon(k) = \frac{k^2}{2m} + \int_{st} \mathcal{U}(s,t,k)$$

Strategy of the Self-energy resummation



Strategy of the Self-energy resummation



Phase-space average approximation: GPS case

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi}\frac{(a_s k_F)}{1 - (a_s k_F/\pi)\frac{9\pi^2}{14}\phi_2(k_F)}$$

Phase-space average approximation: GPS case

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi} \frac{(a_s k_F)}{1 - (a_s k_F/\pi)\frac{9\pi^2}{14}\phi_2(k_F)}$$

$$\mu = \left.\frac{\partial E}{\partial N}\right|_V$$

$$\frac{\mu}{\mu_{FG}} = 1 + \frac{4}{3} \frac{(a_s k_F)}{\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k_F)} + \frac{2}{9} \frac{(a_s k_F)^2 \frac{9\pi^2}{14}\phi_2(k_F)}{\left[\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k_F)\right]^2}$$

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$$\phi_2(k_F) \to \phi_2(k)$$

$$\frac{\epsilon(k)}{\mu_{FG}} = 1 + \frac{4}{3} \frac{(a_s k_F)}{\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k)} + \frac{2}{9} \frac{(a_s k_F)^2 \frac{9\pi^2}{14}\phi_2(k)}{\left[\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k)\right]^2}$$

Phase-space average approximation: GPS case

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$$\mu = \left.\frac{\partial E}{\partial N}\right|_V$$

✓ Lee-Yang Formula

$$\frac{\mu}{\mu_{FG}} = 1 + \frac{4}{3}\frac{(a_s k_F)}{\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k_F)} + \frac{2}{9}\frac{(a_s k_F)^2 \frac{9\pi^2}{14}\phi_2(k_F)}{\left[\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k_F)\right]^2}$$

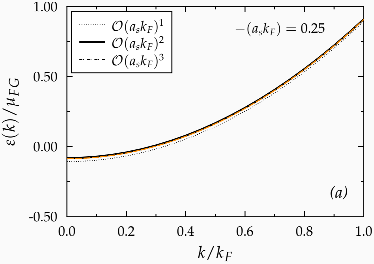
$\phi_2(k_F) \to \phi_2(k)$

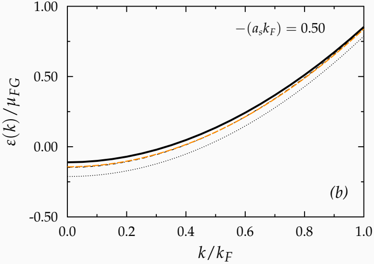
✓ HvH theorem $\mu = \epsilon(k_F)$

$$\frac{\epsilon(k)}{\mu_{FG}} = 1 + \frac{4}{3}\frac{(a_s k_F)}{\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k)} + \frac{2}{9}\frac{(a_s k_F)^2 \frac{9\pi^2}{14}\phi_2(k)}{\left[\pi - (a_s k_F)\frac{9\pi^2}{14}\phi_2(k)\right]^2}$$

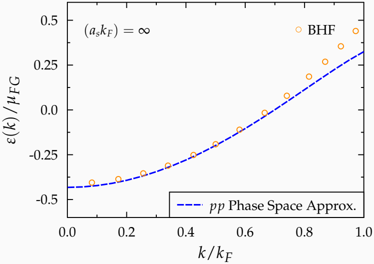
✓ Galitskii Formula

Single particle energy





MBPT: [Platter *et al.*, NPA714 (2003)]
[AB, Lacroix, in preparation]



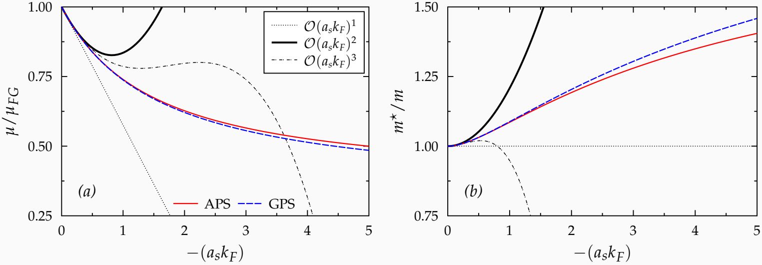
BHF: [Doggen and Kinnumen, Nature (2015)]

- ✓ exact expansion up to $(a_s k_F)^2$ $\rightarrow$ Galitskii formula
- ✓ pathologies removed for $a_s k_F \gg 1$ (more predictive)
- ✓ simpler function of the density

Chemical potential and effective mass

$\mu = \epsilon(k_F)$

$$\frac{m}{m^\star} = \frac{m}{k_F}\frac{\partial \epsilon_k}{\partial k}\bigg|_{k_F}$$



MBPT: [Platter *et al.*, NPA714 (2003)]
[AB, Lacroix, in preparation]

✓ Expansion valid up to $(a_s k_F)^2$ → Galitskii formula

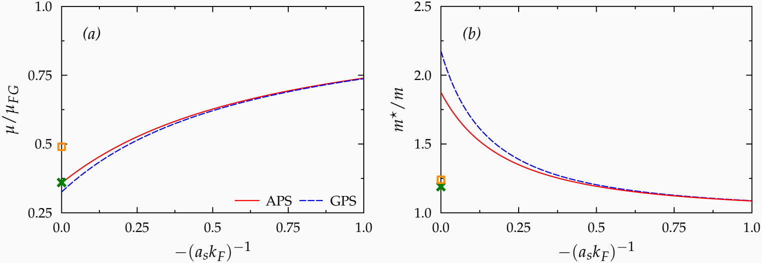
✓ Simple and explicit dependence in density

✓ Finite limit at Unitarity

Discussion

$\mu = \epsilon(k_F)$

$$\frac{m}{m^\star} = \frac{m}{k_F}\frac{\partial \epsilon_k}{\partial k}\bigg|_{k_F}$$



(at unitarity)	superfluid × (expected)	normal □ (expected)	GPS	APS
μ/μ_{FG}	0.37	0.49	0.32	0.36
$m^\star/m$	1.19	1.24	2.18	1.88

Summary

- Ladder (pp or pp/hh) resummation from E to $\Sigma^\star(k)$
  - ✗ quite complex density dependence
  - ✗ strong dependence on the selected diagrams
- **Phase-space approximation of the energy**
  - ✓ **simple and explicit density dependence**
  - ✓ **predictive from low density to unitarity without adjustment**
- **Phase-space approximation of the self-energy**
  - ✓ **simple and explicit density dependence**
  - ✓ **predictive at low and intermediate density**
  - ✗ Unitary limit far from expected results: need to be adjusted
  - ✗ Pairing effect: from normal to superfluid

Perspective

Make explicit the link with density functional theory

$\rightarrow$ *apply to finite systems*

Outlooks and short/long time projects

Ultracold atom physics

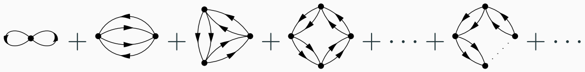
- Extension of the approach
- Include pairing effect
- Study the BEC-BCS crossover (dimer, temperature, etc.)
- Application to Quantum Droplet (statics and dynamics)

Nuclear physics

- Include effective range effect and extend to other partial waves
- Add contributions of the three-body interaction
- Extend the theory to symmetric matter (clustering, etc.)
- Application to nuclei (statics and dynamics)

Basics of diagrammatic framework: ladder approximation I

$E_{int} =$



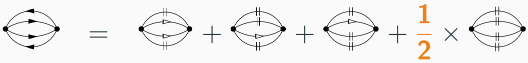
In-medium propagator [Reorganization of many-body diagrams]

$$G(\omega, k) = \frac{1}{\omega - \epsilon_k + i\eta} + 2i\pi\delta(\omega - \epsilon_k)n_k$$

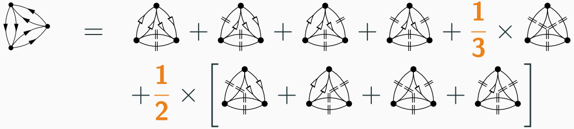
✓ Well adapted to resummation for energy

⚠ Non-trivial symetry factor [Kaiser, NPA **860** (2011)]

$E_{(2)} =$

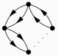


$E_{(3)} =$



Ladder approximation for the energy

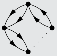
$$E_{(n)} =$$



$$= \frac{8}{m\pi^2}\left(\frac{a_s k_F}{\pi}\right)^n \int_0^{k_F} s^2 ds \int\limits_0^{\sqrt{k_F^2 - s^2}} tdt \; \mathcal{E}_{(n)}(s,t)$$

Resummation of contributions

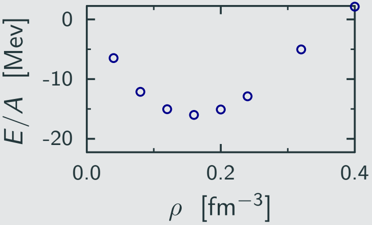
$$E_{int} = \sum_{n=1}^{\infty}$$



$$= \frac{8}{m\pi^2} \int_0^{k_F} s^2 ds \int\limits_0^{\sqrt{k_F - s^2}} tdt \underbrace{\left[\sum_{n=1}^{\infty}\left(\frac{a_s k_F}{\pi}\right)^n \mathcal{E}_{(n)}(s,t)\right]}_{= \, \mathcal{E}(s,t;a_s k_F)}$$

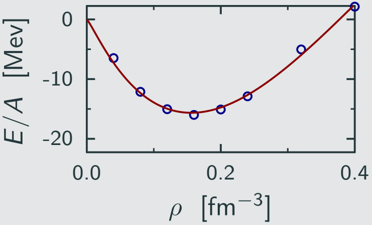
Empirical Density Functional approach: simplest illustration

EoS of sym. nucl. matter



Empirical Density Functional approach: simplest illustration

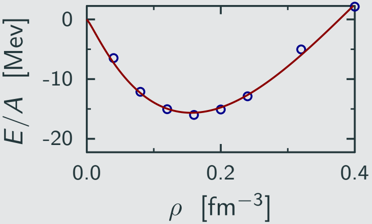
EoS of sym. nucl. matter



- **fit** with polynomial form (5 parameters)

Empirical Density Functional approach: simplest illustration

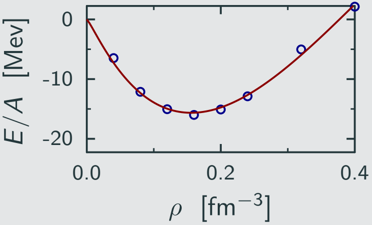
EoS of sym. nucl. matter



- **fit** with polynomial form (5 parameters)
- contains many-body physics: complex **correlation** (beyond Hartree-Fock)

Empirical Density Functional approach: simplest illustration

EoS of sym. nucl. matter



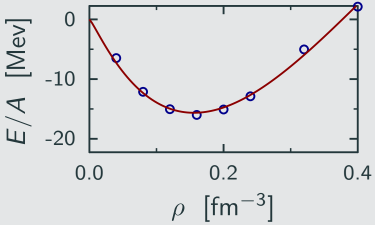
- **fit** with polynomial form (5 parameters)
- contains many-body physics: complex **correlation** (beyond Hartree-Fock)

Unified description

- ✓ ground state – **structure** of atomic nuclei
- ✓ small and large amplitude **dynamics**
- ✓ nuclear **thermodynamic** (finite or infinite systems)

Empirical Density Functional approach: simplest illustration

EoS of sym. nucl. matter



- **fit** with polynomial form (5 parameters)
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Unified description

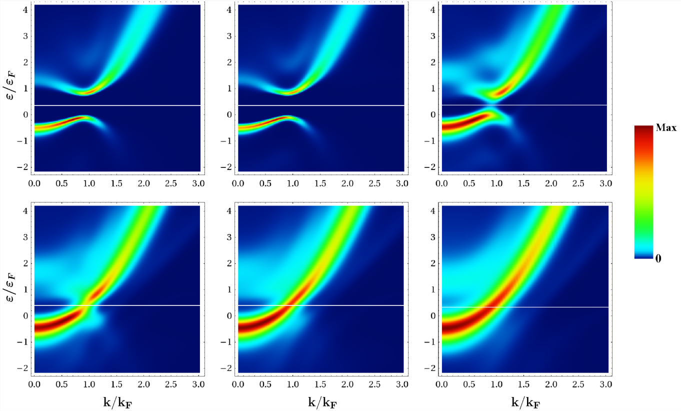
✓ ground state – **structure** of atomic nuclei

✓ small and large amplitude **dynamics**

✓ nuclear **thermodynamic** (finite or infinite systems)

Empirical approach

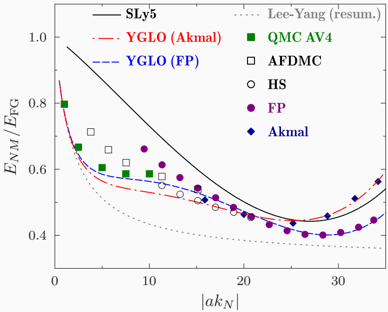
✗ *No direct link with the nuclear interaction*



[Haussmann *et al.*, PRA80 (2009)]

Application for nuclear matter: YGLO functional

$$\frac{E}{A} = K + \frac{B\rho}{1 - R\rho^{1/3} + C\rho^{2/3}} + D\rho^{5/3} + F\rho^{\alpha+1}$$



- $B, R \rightarrow$ Lee–Yang
  - ✓ low density
- C, D, F : higher correlation (fit)

[Yang, Grasso, Lacroix, PRC **94** (2016)]

[Yang, Grasso, Lacroix, PRC **94** (2016)]

Application for system close to unitarity: effective range

$$\frac{E}{E_{FG}} = 1 + \frac{(a_s k_F) U_0}{1 - (a_s k_F) U_1} + \frac{(r_s k_F) R_0 / \left[1 - (a_s k_F)^{-1} R_1\right]}{\left[1 - (a_s k_F)^{-1} R_1 + (r_s k_F) R_2\right]}$$

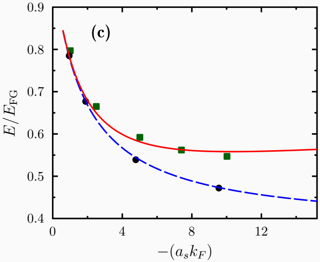
Low density fixe U_0, R_1

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi}(a_s k_F) + \Big[\underbrace{\frac{4}{21\pi^2}(11 - 2\ln 2)}_{\text{relaxed constraint}} + \frac{1}{6\pi}(r_s k_F) \Big] (a_s k_F)^2 + \cdots$$

Unitarity $(r_s k_F \ll 1)$ fixe U_1, R_0, R_2

$$\frac{E}{E_{FG}} = \xi_0 + (r_s k_F)\eta_e + (r_s k_F)^2 \delta_e + \cdots$$

[Lacroix, AB, *et al.*, PRC **95** (2017)]



Lee-Yang energy density functional for finite systems

Local Density Approximation (LDA)

$$E(\rho) \underset{\rho \to \rho(\boldsymbol{r})}{\longmapsto} E[\rho] = \int d^3 r \mathcal{E}(\rho(\boldsymbol{r}))$$

✓ Quantum Droplet (other refs.)

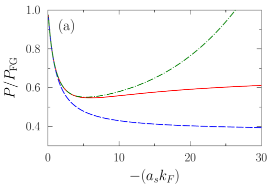
[Bonnard, Grasso, Lacroix, PRC **98** (2018)]

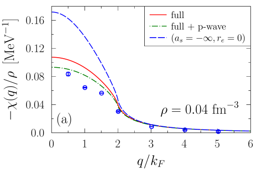
Application for system close to unitarity: some recent results

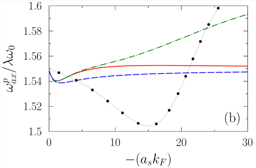
Motivated by neutron matter close to unitarity regime: large s-wave scattering length. At saturation $|a_s k_F| \sim 20$

- Thermodynamical properties: pressure, compressibility, sound velocity, chemical potential, . . .
  $P = \rho^2 \left. \frac{\partial E/N}{\partial \rho} \right|_N$
- Linear (static) response:
  $\delta\rho(\boldsymbol{r}) = \int d^3r' \chi(\boldsymbol{r} - \boldsymbol{r'}) V_{ext}(\boldsymbol{r'})$
- Collective modes in hydrodynamical regime for trapped Fermi systems

Finite limit at large density Better than Skyrme at low density [AB, Lacroix, PRC **97** (2018)]







+put skyrme in all panel

Application for system close to unitarity

$$\frac{E}{E_{FG}} = 1 + \frac{(a_s k_F) U_0}{1 - (a_s k_F) U_1}$$

[Lacroix, PRA **94** (2016)]

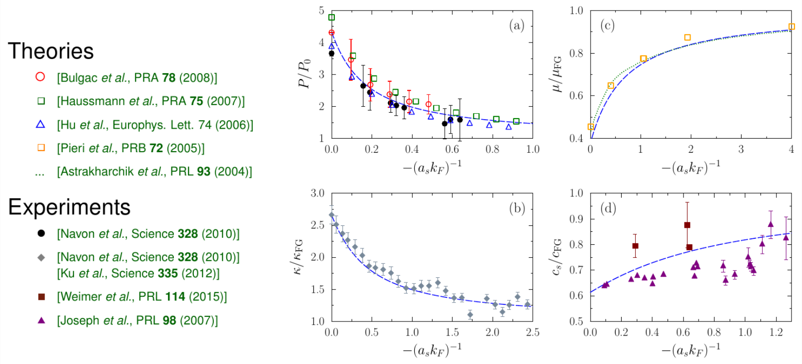
✗ At low density (Lee-Yang):

$$\frac{E}{E_{FG}} = 1 + \frac{10}{9\pi}(a_s k_F) + \underbrace{\frac{4}{21\pi^2}(11 - 2\ln 2)(a_s k_F)^2}_{\textit{relaxed constraint}} + \cdots$$

✓ Unitary limit imposed

$$\frac{E}{E_{FG}} = \xi_0 \simeq 0.37 \quad \longmapsto \quad U_1 = \frac{9\pi}{10}(1 - \xi_0)^{-1}$$

Application for system close to unitarity



[AB, Lacroix, PRC **97** (2018)]

In general the non-empirical DFT works very well in cold atoms at unitarity and away from unitarity.

Ladder approximation I: semi-analytical results

Combined pp and hh ladder resummation $\Sigma^\star(k) = U(k) + iW(k)$

$$U(k<k_F) = 16(g-1)\frac{k_F^2}{2m}\int_0^1 s^2 ds \int_0^{\sqrt{1-s^2}} tdt \left\{ \frac{(a_s k_F)^2 I_*(s,t)\widehat{R}(s,t,p) + (a_s k_F)\widehat{I_*}(s,t,p)\left[\pi - (a_s k_F)R(s,t)\right]}{\left[\pi - (a_s k_F)R(s,t)\right]^2 + \left[(a_s k_F)\pi I(s,t)\right]^2} - \widehat{R}(s,t,p)\delta\left(\frac{\pi}{a_s k_F} - R(s,t)\right)\Theta(1-s^2-t^2)\right\}$$

$$W(k<k_F) = 16(g-1)\frac{k_F^2}{2m}\int_0^1 s^2 ds \int_0^{\sqrt{1-s^2}} tdt \; \frac{(a_s k_F)^2 \pi\left[\widehat{I_*}(s,t,p) - \widehat{I}(s,t,p)\right] I_*(s,t)}{\left[\pi - (a_s k_F)R(s,t)\right]^2 + \left[(a_s k_F)\pi I(s,t)\right]^2}$$

$$U(k>k_F) = +16(g-1)\frac{k_F^2}{2m}\int_0^{(1+p)/2} s^2 ds \int_0^{(1+p)/2} tdt \left\{ \frac{(a_s k_F)^2 I_*(s,t)\widehat{R}(s,t,p) + (a_s k_F)\widehat{I_*}(s,t,p)\left[\pi - (a_s k_F)R(s,t)\right]}{\left[\pi - (a_s k_F)R(s,t)\right]^2 + \left[(a_s k_F)\pi I(s,t)\right]^2} - \widehat{R}(s,t,p)\delta\left(\frac{\pi}{a_s k_F} - R(s,t)\right)\Theta(1-s^2-t^2)\right\}$$

$$W(k>k_F) = -16(g-1)\frac{k_F^2}{2m}\int_0^{(1+p)/2} s^2 ds \int_0^{(1+p)/2} tdt \; \frac{(a_s k_F)^2 \pi \widehat{I_*}(s,t,p) I(s,t)\Theta(s^2+t^2-1)}{\left[\pi - (a_s k_F)R(s,t)\right]^2 + \left[(a_s k_F)\pi I(s,t)\right]^2}$$

Only pp ladder resummation $\Sigma^\star_{pp}(k) = U_{pp}(k) + iW_{pp}(k)$

$$U_{pp}(k<k_F) = 16(g-1)\frac{k_F^2}{2m}\int_0^1 s^2 ds \int_0^{\sqrt{1-s^2}} tdt \left\{ \frac{(a_s k_F)^2 \widehat{F}(s,t,p) I_*(s,t)}{\left[\pi - (a_s k_F)F(s,t)\right]^2} - \frac{(a_s k_F)\widehat{I_*}(s,t,p)}{\pi - (a_s k_F)F(s,t)}\right\}$$

$$W_{pp}(k<k_F) = 16(g-1)\frac{k_F^2}{2m}\int_0^1 s^2 ds \int_0^{\sqrt{1-s^2}} tdt \left\{ \frac{(a_s k_F)^2\left[\widehat{I_*}(s,t,p) - \widehat{I}(s,t,p)\right] I_*(s,t)}{\left[\pi - (a_s k_F)F(s,t)\right]^2}\right\}$$

$$U_{pp}(k>k_F) = +16(g-1)\frac{k_F^2}{2m}\int_0^{(1+p)/2} s^2 ds \int_0^{(1+p)/2} tdt \left\{ \frac{(a_s k_F)^2 \widehat{F}(s,t,p) I_*(s,t)}{\left[\pi - (a_s k_F)F(s,t)\right]^2} - \frac{(a_s k_F)\widehat{I_*}(s,t,p)\Theta(1-s^2-t^2)}{\pi - (a_s k_F)F(s,t)} - \frac{(a_s k_F)\left[\pi - (a_s k_F)F(s,t)\right]\widehat{I_*}(s,t,p)\Theta(s^2+t^2-1)}{\left[\pi - (a_s k_F)F(s,t)\right]^2 + \left[(a_s k_F)\pi I(s,t)\right]^2}\right\}$$

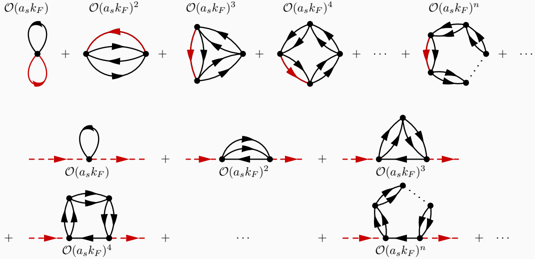
$$W_{pp}(k>k_F) = -16(g-1)\frac{k_F^2}{2m}\int_0^{(1+p)/2} s^2 ds \int_0^{(1+p)/2} tdt \left\{ \frac{(a_s k_F)^2 \widehat{I_*}(s,t,p) I(s,t)\Theta(s^2+t^2-1)}{\left[\pi - (a_s k_F)F(s,t)\right]^2 + \left[(a_s k_F)\pi I(s,t)\right]^2}\right\}$$

$+$ one more slide to define analytic functions
$I(s,t), I_*(s,t), \widehat{I}(s,t,p), \widehat{I_*}(s,t,p), F(s,t), R(s,t), \widehat{F}(s,t,p), \widehat{R}(s,t,p)$

Self-energy and Quasi-particle properties

$E_{int} =$

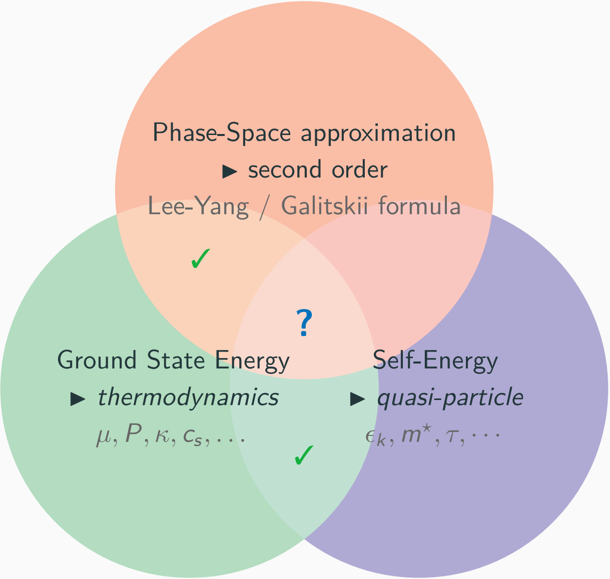
$\Sigma^\star(k) =$



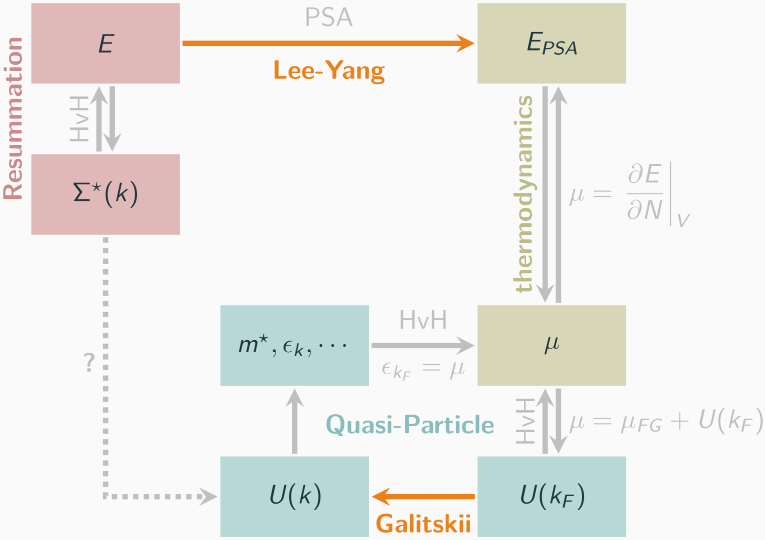
In-medium insertion approach

- ✓ Well adapted to resummation
- ✓ Direct link with Landau Theory of Fermi Liquid

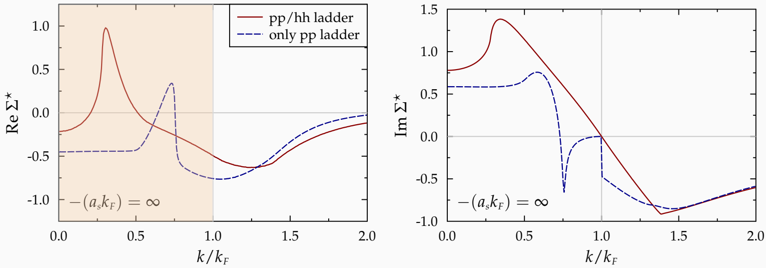
Strategy of the Self-energy resummation I



Summary of the strategy for the single-particle potential



Ladder approximation II: numerical results



- ✓ correct limit at $a_s k_F \ll 1$ (Galitskii expansion)
- ✓ finite limit at unitarity
- ✗ Strong dependence of retained diagrams
- ✗ Unpractical to link with DFT

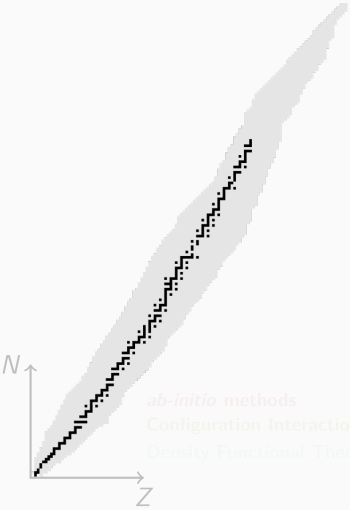
Nuclear Theory Landscape

Degrees of Freedom

QCD

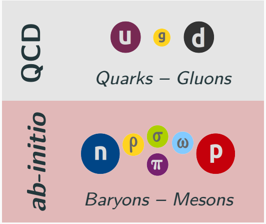
u g d

Quarks – Gluons

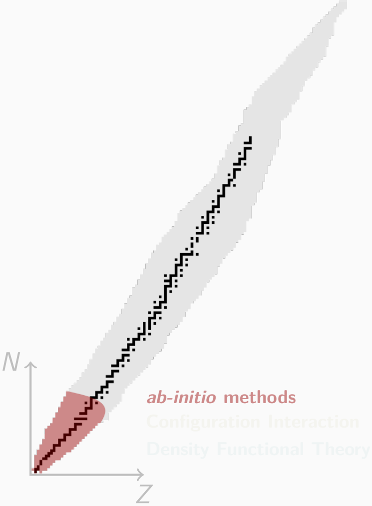


Nuclear Theory Landscape

Degrees of Freedom

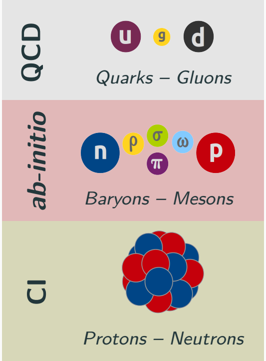


fit to few-body experiments

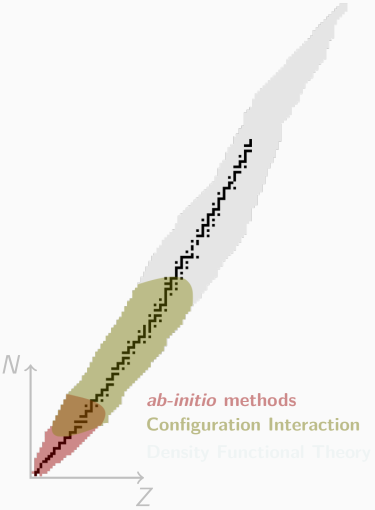


Nuclear Theory Landscape

Degrees of Freedom

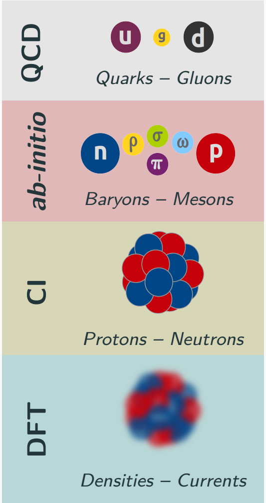


fit to few-body experiments



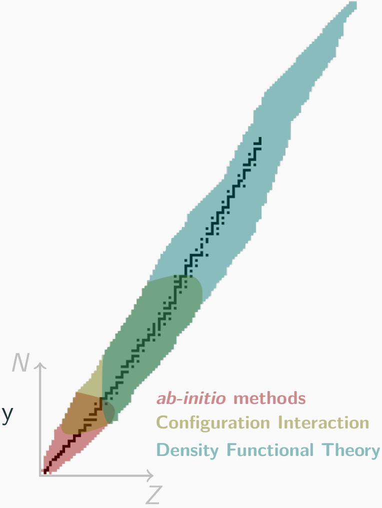
Nuclear Theory Landscape

Degrees of Freedom



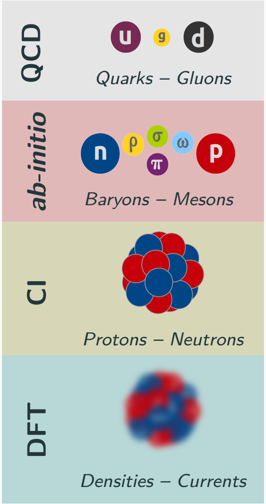
fit to few-body experiments

fit to many-body experiments



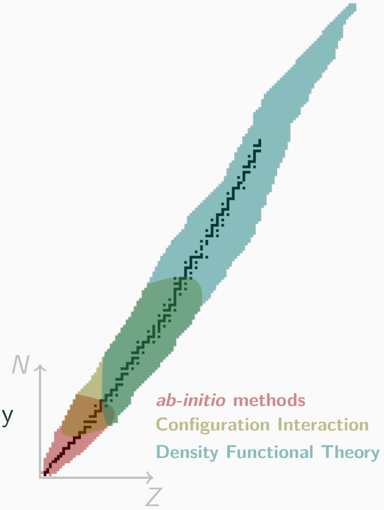
Nuclear Theory Landscape

Degrees of Freedom



fit to few-body experiments

fit to many-body experiments



Ladder approximation I: semi-analytical results

Combined pp and hh ladder resummation $\Sigma^\star(k) = U(k) + iW(k)$

$$U(k<k_F) = \frac{8}{mk_F^3}\int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2-s^2}} t dt$$

$$\left\{\frac{(a_s k_F)^2 I_*(s,t)\widehat{R}(s,t,k) + (a_s k_F)\widehat{I_*}(s,t,k)\left[\pi - (a_s k_F)R(s,t)\right]}{\left[\pi - (a_s k_F)R(s,t)\right]^2 + \left[(a_s k_F)\pi I(s,t)\right]^2}\right.$$

$$\left. - \widehat{R}(s,t,k)\delta\left(\frac{\pi}{a_s k_F} - R(s,t)\right)\Theta(k_F^2 - s^2 - t^2)\right\}$$

Only pp ladder resummation $\Sigma^\star_{pp}(k) = U_{pp}(k) + iW_{pp}(k)$

$$U_{pp}(k<k_F) = \frac{8}{mk_F^3}\int_0^{k_F} s^2 ds \int_0^{\sqrt{k_F^2-s^2}} t dt$$

$$\left\{\frac{(a_s k_F)^2 \widehat{F}(s,t,k) I_*(s,t)}{\left[\pi - (a_s k_F)F(s,t)\right]^2} - \frac{(a_s k_F)\widehat{I_*}(s,t,k)}{\pi - (a_s k_F)F(s,t)}\right\}$$

[Kaiser, EPJA **49** (2013)]

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